Categorifying L-functions

Andrew J. Kobin

ajkobin@emory.edu

Algebraic and Analytic Theory of Elliptic Curves

October 1, 2022

Joint work with Jon Aycock

Introduction

Based on applications of

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments:	23 pages; minor changes and additional references added
Subjects:	**Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes:	11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as:	arXiv:2011.13903 **[math.NT]**
	(or arXiv:2011.13903v2 **[math.NT]** for this version)

and

Categorifying quadratic zeta functions

Jon Aycock, Andrew Kobin

The Dedekind zeta function of a quadratic number field factors as a product of the Riemann zeta function and the L-function of a quadratic Dirichlet character. We categorify this formula using objective linear algebra in the abstract incidence algebra of the division poset.

Comments:	27 pages
Subjects:	**Number Theory (math.NT)**
MSC classes:	11M06, 11M41, 18N50, 06A11, 16T10
Cite as:	arXiv:2205.06298 **[math.NT]**
	(or arXiv:2205.06298v1 **[math.NT]** for this version)

to L-functions of elliptic curves (work in progress with J. Aycock)

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

motivic L-functions
$Z_{mot}(E, t)$

arithmetic L-functions
$L(E, s)$

local L-functions
$Z(E/\mathbb{F}_q, t)$

Introduction

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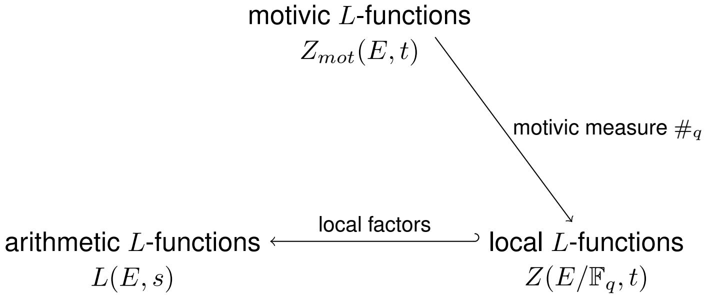

Varieties over Finite Fields

Let X be an algebraic variety over $\mathbb{F}_q$. Its point-counting zeta function is the power series

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

Historically, this is called a zeta function because it has:

- a product formula $Z(X,t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}$
- a functional equation
- an expression as a *rational function*
- a Riemann hypothesis which is a theorem!

Varieties over Finite Fields

We can formalize certain properties of $Z(X,t)$ in an algebra of "arithmetic functions".

Let $Z_0^{\text{eff}}(X)$ be the set of effective 0-cycles on X, i.e. formal $\mathbb{N}_0$-linear combinations of closed points of X, written $\alpha = \sum m_x x$.

We say $\beta \leq \alpha$ if $\beta = \sum n_x x$ with $n_x \leq m_x$ for all $x \in |X|$.

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

We call the distinguished element $\zeta : \alpha \mapsto 1$ the *zeta function* of X.

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The degree map $Z_0^{\mathrm{eff}}(X) \to Z_0^{\mathrm{eff}}(\operatorname{Spec} \mathbb{F}_q) \cong \mathbb{Z}$ determines an algebra map

$$\begin{aligned} A_X &\longrightarrow A_{\operatorname{Spec} \mathbb{F}_q} \cong \mathbb{C}[[t]] \\ & \qquad f \leftrightarrow \sum_{n=0}^{\infty} f(n) t^n \\ f &\longmapsto \left(\text{“}\deg_*(f)\text{”} : n \mapsto \sum_{\deg(\alpha)=n} f(\alpha) \right) \\ \zeta &\longmapsto \text{“}\deg_*(\zeta)\text{”} \leftrightarrow Z(X,t) \end{aligned}$$

What’s really going on?

What's really going on?

These A_X are examples of **reduced incidence algebras**, which come from a much more general simplicial framework.

Numerical Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} S_1 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} S_2 \cdots .$$

Example

A poset $\mathcal{P}$ determines a simplicial set $N\mathcal{P}$ with:

- 0-simplices $=$ elements $x \in \mathcal{P}$
- 1-simplices $=$ intervals $[x, y]$
- 2-simplices $=$ decompositions $[x, y] = [x, z] \cup [z, y]$
- etc.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; S_2 \cdots .$$

Example

More generally, any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

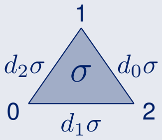
Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence coalgebra** of a decomposition set S is the free k-vector space $C(S) = \bigoplus_{x \in S_1} kx$ with comultiplication

$$\begin{aligned} C(S) &\longrightarrow C(S) \otimes C(S) \\ x &\longmapsto \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma. \end{aligned}$$



Numerical Incidence Algebras

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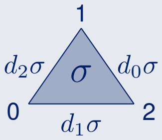
Definition

The **numerical incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication

$$I(S) \otimes I(S) \longrightarrow I(S)$$
$$f \otimes g \longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma)g(d_0\sigma).$$



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The **numerical incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication $f * g$.

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** ("linear algebra with sets"):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

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$\longleftarrow$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Abstract Incidence Algebras

How do we construct $I(S)$ as an “objective vector space”?

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So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{array}{ccccc} & & M & & \\ & \overset{s}{\swarrow} & & \overset{t}{\searrow} & \\ S_1 & & & & * \end{array} \right)$$

Abstract Incidence Algebras

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$$f = \left(S_1 \xleftarrow{\;s\;} M \xrightarrow{\;t\;} * \right)$$

Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(S_1 \xleftarrow{\;\mathrm{id}\;} S_1 \longrightarrow * \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \left(S_1 \xleftarrow{\;s\;} M \longrightarrow * \right) \quad \text{and} \quad g = \left(S_1 \xleftarrow{\;t\;} N \longrightarrow * \right)$$

the convolution $f * g \in I(S)$ is represented by

$$(f * g) = \left(\begin{array}{ccccc} & & P & & \\ & \swarrow & & \searrow & \\ & S_2 & & M \times N & \\ {\scriptstyle d_1} \swarrow & & \searrow {\scriptstyle (d_2, d_0)} \quad {\scriptstyle s \times t} \swarrow & & \searrow \\ S_1 & & S_1 \times S_1 & & * \end{array} \right)$$

Elliptic Curves

For an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

Theorem (Aycock–K., '22+ϵ)

In the reduced incidence algebra $\widetilde{I}(E) := \widetilde{I}(Z_0^{\mathrm{eff}}(E))$, there is an equivalence of linear functors

$$\pi_* \zeta_E + \zeta_{\mathbb{P}^1} * L(E)^- \cong \zeta_{\mathbb{P}^1} * L(E)^+$$

where $\pi : E \to \mathbb{P}^1$ is a fixed double cover and $L(E)^+$ and $L(E)^-$ are functors in $\widetilde{I}(\mathbb{P}^1)$.

Pushing forward to $\widetilde{I}(\operatorname{Spec} \mathbb{F}_q)$ and taking cardinalities, it reads

$$\pi_* \zeta_E = \pi_* \zeta_{\mathbb{P}^1} * (L(E)^+ - L(E)^-) = \pi_* \zeta_{\mathbb{P}^1} * L(E).$$

Motivic Zeta Functions

For any k-variety X, $Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$ decategorifies to other zeta functions by applying motivic measures (point counting, Euler characteristic, etc.)

Das–Howe ('21) lift $Z_{mot}(X,t)$ to a numerical incidence algebra

$$\widetilde{I}_{mot}(\Gamma^{\bullet,+}(X)) = \prod_{n=0}^{\infty} K_0(\mathtt{Var}_{/\Gamma^n X})$$

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Idea (in progress): lift $Z_{mot}(X,t)$ to an objective incidence algebra $I(\Gamma^{\bullet,+}(X))$ in the category of simplicial k-varieties. Passing to K_0 recovers Das and Howe's construction.

Highlights and Dreams for the Future

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- General: most* zeta functions can be produced this way
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- It's pretty fun to prove things!

Future work:

- Construct $\zeta_{\mathcal{X}}$ for an algebraic stack $\mathcal{X}$
- Lift L-functions of representations $L(V)$
- Archimedean zeta functions.

Thank you!