Non-orientable enumerative problems in $\mathbb{A}^1$-homotopy theory

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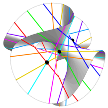
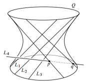
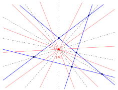
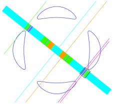
Algebra & Number Theory Seminar

November 16, 2021

EMORY UNIVERSITY

Joint work with Libby Taylor

Introduction

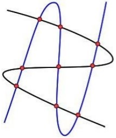
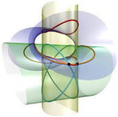
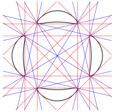





Goal of $\mathbb{A}^1$-enumerative geometry: count geometric objects over a field k $(\mathrm{char}\, k \neq 2)$ when the count is fixed over $\bar{k}$.

This can be done using Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory (more on that later).

"Enriched counts" take values in the Grothendieck–Witt ring of quadratic forms:

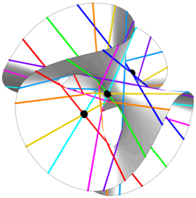
$GW(k) =$ group completion of $\{$nondegen. symm. bilinear forms $/k\}$.

Introduction

Example (Lines on a smooth cubic)

The Cayley–Salmon Theorem says that for a smooth cubic surface $X/\mathbb{C}$, there are exactly $\mathbf{27}$ lines on X.



Over other fields, this count is not fixed, e.g. over $\mathbb{R}$, there can be 3, 7, 15 or 27 lines on X. However, there is a "signed count" which is fixed:

$$\#\text{real hyperbolic lines on } X - \#\text{real elliptic lines on } X = \mathbf{3}.$$

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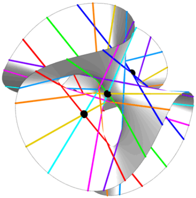


(Kass–Wickelgren '17) The lines on a smooth cubic surface X/k can be enumerated in $GW(k)$ by the class

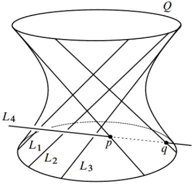
$$\mathbf{15\langle 1\rangle + 12\langle -1\rangle}$$

where $\langle a\rangle$ is the class of the quadratic form $q(x) = ax^2$.

Introduction

Example (Lines in $\mathbb{P}^3$)

Over $\mathbb{C}$, there are **2** lines meeting 4 general lines in $\mathbb{P}^3$.



(Srinivasan–Wickelgren '21) The lines meeting 4 general lines in $\mathbb{P}^3$ can be enumerated in $GW(k)$ by the class

$$\langle \mathbf{1} \rangle + \langle -\mathbf{1} \rangle$$

Introduction

Example (Lines in $\mathbb{P}^n$)

More generally, over $\mathbb{C}$ there are $\mathbf{c(n-1)}$ lines intersecting $2n-2$ general codimension 2 hyperplanes in $\mathbb{P}^n$ when n is odd:

$$c(n-1) = \frac{(2n-2)!}{n!(n-1)!} \text{Catalan numbers}$$

(Srinivasan–Wickelgren '21) The lines meeting $2n-2$ general codimension 2 hyperplanes in $\mathbb{P}^n$, n odd, can be enumerated in $GW(k)$ by the class

$$\frac{\mathbf{c(n-1)}}{\mathbf{2}}\langle \mathbf{1}\rangle + \langle \mathbf{-1}\rangle$$

Introduction

Other enumerative problems that have solutions in $GW(k)$ include:

- (Larson–Vogt '19) $\mathbf{16\langle 1\rangle + 12\langle -1\rangle}$ bitangents to a smooth plane quartic
- (McKean '20) Arithmetic Bézout's Theorem: the intersection of n general hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \ldots, d_n$ is enumerated by $\mathbf{\frac{d_1 \cdots d_n}{2}(\langle 1\rangle + \langle -1\rangle)}$
- (Pauli '20) $\mathbb{A}^1$-enumerative version of Milnor numbers
- (Brazelton–McKean–Pauli '21) $\mathbb{A}^1$-Euler characteristics of Grassmannians
- (Kim–Park '21) $\mathbb{A}^1$-degrees of covers of modular curves
- (Bachmann–Wickelgren '21) $\mathbf{160839\langle 1\rangle + 160650\langle -1\rangle}$ dimension 3 hyperplanes in a 7-dimensional cubic hypersurface (and generalizations)

Introduction

These results* require an *orientation* on the vector bundle used to enumerate the geometric objects.

Libby Taylor and I extend these techniques to *non-orientable vector bundles* (and associated non-orientable enumerative problems) using algebraic stacks.

Mathematics > Algebraic Geometry

[Submitted on 14 Nov 2019 (v1), last revised 18 Aug 2020 (this version, v4)]

$\mathbb{A}^1$-Local Degree via Stacks

Andrew Kobin, Libby Taylor

We extend results of Kass--Wickelgren to define an Euler class for a non-orientable (or non-relatively orientable) vector bundle on a smooth scheme, valued in the Grothendieck--Witt group of the ground field. We use a root stack construction to produce this Euler class and discuss its relation to other versions of an Euler class in $\mathbb{A}^1$-homotopy theory. This allows one to apply Kass--Wickelgren's technique for arithmetic enrichments of enumerative geometry to a larger class of problems; as an example, we use our construction to give an arithmetic count of the number of lines meeting 6 planes in $\mathbb{P}^4$.

Outline of the talk

- **Introduction**
- The Grothendieck–Witt ring
- $\mathbb{A}^1$-local degree
- $\mathbb{A}^1$-Euler classes of vector bundles
- Non-oriented enumerative problems

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- **The Grothendieck–Witt ring**
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The Grothendieck–Witt ring

Let k be a field of characteristic $\neq 2$. The **Grothendieck–Witt ring** of k is the group completion $GW(k)$ of

$$\{\text{nondeg. symm. bilinear forms on } k\}/\text{iso.} \quad \text{under } \oplus, \otimes.$$

An isomorphism class is represented by a bilinear form $b : V \times V \to k$ or equivalently a quadratic form $f(x) = b(x,x)$, e.g.

$$\begin{aligned}
(x,y) &\mapsto x \cdot y & &\longleftrightarrow & q(x) &= ||x||^2 \\
(x,y) &\mapsto x_1y_1 - x_2y_2 & &\longleftrightarrow & q(x) &= x_1^2 - x_2^2 \\
(x,y) &\mapsto x_1y_1 - x_2y_2 - x_3y_3 & &\longleftrightarrow & q(x) &= x_1^2 - x_2^2 - x_3^2
\end{aligned}$$

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$GW(k)$ is generated by symbols $\langle a \rangle$ for $a \in k^\times/k^{\times 2}$, denoting the iso. class of the rank 1 bilinear form $(x, y) \mapsto axy$, satisfying:

1. $\langle a \rangle \langle b \rangle = \langle ab \rangle$
2. $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$
3. $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$, the *hyperbolic form* $x^2 - y^2$

The Grothendieck–Witt ring

Example

For $k = \mathbb{C}$, *rank* gives an isomorphism

$$\begin{aligned} GW(\mathbb{C}) &\longrightarrow \mathbb{Z} \\ \langle a \rangle &\longmapsto 1 \end{aligned}$$

The Grothendieck–Witt ring

Example

For $k = \mathbb{R}$, *rank and signature* give an isomorphism

$$\begin{aligned} GW(\mathbb{R}) &\longrightarrow \mathbb{Z} \times \mathbb{Z} \\ \langle a \rangle &\longmapsto \begin{cases} (1,1), & a > 0 \\ (1,-1), & a < 0 \end{cases} \end{aligned}$$

The Grothendieck–Witt ring

Example

For $k = \mathbb{F}_q$, *rank and discriminant* give an isomorphism

$$GW(\mathbb{F}_q) \longrightarrow \mathbb{Z} \times \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$$
$$\langle a \rangle \longmapsto (1, a)$$

The Grothendieck–Witt ring

Key idea: geometric configurations over k can be enumerated by classes in $GW(k)$ and classical solutions (e.g. over $\mathbb{C}$ or $\mathbb{R}$) can be recovered by taking invariants of these classes (e.g. rank, signature, discriminant).

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Example

The $\mathbf{15}\langle\mathbf{1}\rangle + \mathbf{12}\langle\mathbf{-1}\rangle$ lines on a smooth cubic surface become

- (rank) $15 + 12 = \mathbf{27}$ over $k = \mathbb{C}$
- (sign.) $15 - 12 = \mathbf{3}$ over $k = \mathbb{R}$
- (disc.) $15\,\mathrm{disc}\langle 1\rangle + 12\,\mathrm{disc}\langle -1\rangle \equiv \mathbf{0} \pmod{\mathbf{2}}$ over $k = \mathbb{F}_{p^2}$
- etc.

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- **$\mathbb{A}^1$-local degree**
- $\mathbb{A}^1$-Euler classes of vector bundles
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Topological degree

Recall: a continuous map $f : S^n \to S^n$ has degree $\deg(f) \in \mathbb{Z}$ defined by

$$\deg(f) = \sum_{f(x)=y} \deg_x(f)$$

where y is a regular value of f and $\deg_x(f)$ is the *local degree* at x.

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Local degree: in local coordinates about x, f determines a map $(f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ with $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ and

$$\deg_x(f) = \begin{cases} +1, & J(x) > 0 \\ -1, & J(x) < 0 \end{cases}$$

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We can view this as a homomorphism $\deg : [S^n, S^n] \to \mathbb{Z}$.

$\mathbb{A}^1$-topological degree

Observation: as a real algebraic variety, $S^n \cong \mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}$.

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Over arbitrary k, Morel ('06) constructed a map

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} \longrightarrow GW(k)$$

using $\mathbb{A}^1$-*homotopy theory*.

Brief summary: cohomology functors on Sm_k are represented by objects in a category $SH(k)$ and we have

$$[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} = \mathrm{End}_{SH(k)}(\mathbb{P}^n/\mathbb{P}^{n-1}) \quad \text{and} \quad GW(k) \cong \widetilde{CH}^0(k)$$

for the functors $[-, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1}$ and $\widetilde{CH}^0(-) \cong K_0^{MW}(-)$.

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$\mathbb{A}^1$-topological degree

A map $f : \mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^{n-1}$ has $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}(f) \in GW(k)$ defined by

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$$\deg_x^{\mathbb{A}^1}(f) = \langle J(x) \rangle \in GW(k) \quad \text{if } k(x) = k.$$

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$$\deg_x^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(x)/k}\langle J(x)\rangle \in GW(k) \quad \text{in general.}$$

$\mathbb{A}^1$-topological degree

This gives us a way of constructing classes in $GW(k)$:

$$f : \mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^{n-1} \quad \rightsquigarrow \quad \deg^{\mathbb{A}^1}(f) \in GW(k).$$

Next: turn an enumerative problem into such a map f.

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- **$\mathbb{A}^1$-Euler classes of vector bundles**
- Non-oriented enumerative problems

Enumerative problems and Euler classes

Many enumerative problems can be solved by computing the **Euler class** $e(E)$ of a vector bundle $E \to X$.

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Example (Lines on a smooth cubic, revisited)

View our cubic as $S = \{F = 0\} \subseteq \mathbb{P}^3$.

Lines in $\mathbb{P}^3$ are parametrized by the Grassmannian $\mathrm{Gr}(2,4)$.

There is a rank 6 vector bundle $E \to \mathrm{Gr}(2,4)$ such that

$$E_\ell = \{\text{homogeneous cubic forms on } \ell\}.$$

There is also a section $\sigma_F : \mathrm{Gr}(2,4) \to E, \ell \mapsto F|_\ell$, so that

$$\{\text{zeroes of } \sigma_F\} = \{\text{lines } \ell \subset \mathbb{P}^3 \text{ lying on } S\}.$$

Over $\mathbb{C}$, the Euler class $e(E, \sigma_F) \in H^8(\mathrm{Gr}(2,4); \mathbb{Z}) \cong \mathbb{Z}$ is 27.

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****This does not depend on* σ_F***

Enumerative problems and Euler classes

Recall: For an *oriented* rank r vector bundle $E \to X$ with section $\sigma \in H^0(X, E)$, the topological Euler class $e(E, \sigma)$ is a characteristic class in $H^r(X; \mathbb{Z})$.

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When $r = \dim X$, $H^r(X; \mathbb{Z}) \cong \mathbb{Z}$ and $e(E, \sigma)$ can be computed by

$$e(E, \sigma) = \sum_{\sigma(x)=0} \mathrm{ind}_x(\sigma)$$

where $\mathrm{ind}_x(\sigma)$ is the local index of σ at x:

- in local coordinates around x, σ looks like a map $\mathbb{R}^r \to \mathbb{R}^r$
- $\mathrm{ind}_x(\sigma)$ is the degree of the bottom map

$$\begin{array}{ccc} X/(X \setminus \{x\}) & \longrightarrow & E/(E \setminus \{\sigma(x)\}) \\ \uparrow & & \uparrow \\ S^r \cong \mathbb{R}^r/(\mathbb{R}^r \setminus \{0\}) & \longrightarrow & \mathbb{R}^r/(\mathbb{R}^r \setminus \{0\}) \cong S^r \end{array}$$

$\mathbb{A}^1$-Euler classes

For an *oriented* rank r vector bundle $E \to X$ with section $\sigma \in H^0(X, E)$, the $\mathbb{A}^1$-Euler class $e(E, \sigma)$ is a class in $\widetilde{CH}^r(X, \det E^\vee)$.

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$$e(E, \sigma) = \sum_{\sigma(x)=0} \mathrm{ind}_x(\sigma)$$

where $\mathrm{ind}_x(\sigma)$ is the $\mathbb{A}^1$-local index of σ at x:

- in Nisnevich local coordinates around x, σ looks like $\mathbb{A}^r \to \mathbb{A}^r$
- $\mathrm{ind}_x(\sigma)$ is the $\mathbb{A}^1$-degree of the bottom map

$$\begin{array}{ccccc} & X/(X \setminus \{x\}) & \longrightarrow & E/(E \setminus \{\sigma(x)\}) & \\ & \uparrow & & \uparrow & \\ \mathbb{P}^r/\mathbb{P}^{r-1} \cong & \mathbb{A}^r/(\mathbb{A}^r \setminus \{0\}) & \longrightarrow & \mathbb{A}^r/(\mathbb{A}^r \setminus \{0\}) & \cong \mathbb{P}^r/\mathbb{P}^{r-1} \end{array}$$

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Orientability

For $e(E, \sigma)$ to be defined in $GW(k)$ and to be independent of σ (also, choices of coordinates, etc.), E must be oriented.

This is equivalent to $\det E^\vee \cong L^{\otimes 2}$ for some line bundle $L \to X$. An **orientation** of E is a choice of section $s \in H^0(X, \det E^\vee)$ which is a square.

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Unfortunately, many enumerative problems do not produce orientable vector bundles, e.g.

- Bitangents to a smooth plane quartic
- Some cases of Bézout's Theorem
- Lines meeting $2n-2$ general codim. 2 hyperplanes in $\mathbb{P}^n$ for n even
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Non-orientable enumerative problems

Suppose (E, σ) is a vector bundle and section over X that represents a non-orientable enumerative problem, so $L = \det E^\vee$ is not a square.

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Naive solution: Take a double cover $Y \xrightarrow{\pi} X$, pull (E,σ) back to $(\pi^*E, \pi^*\sigma)$ and compute $e(\pi^*E, \pi^*\sigma)$.

$$\begin{array}{cccc} & Y \longrightarrow X & \\ \text{orientable} & (\pi^*E, \pi^*\sigma) \longleftarrow (E,\sigma) & \text{non-orientable} \end{array}$$

In general, this depends on π (and possibly σ, the orientation, etc.)

Non-orientable enumerative problems

Suppose (E, σ) is a vector bundle and section over X that represents a non-orientable enumerative problem, so $L = \det E^\vee$ is not a square.

Our solution: Let $\mathcal{X} = \sqrt{(L,s)/X}$ be the **root stack** of X with respect to L and an appropriate section $s \in H^0(X, L)$.

$$\mathcal{X} \longrightarrow X$$

orientable $\quad (\mathcal{E}, \tau) \longleftarrow (E, \sigma) \quad$ non-orientable

Theorem

There is a well-defined Euler class $e(\mathcal{E}, \tau) \in GW(k)$ which is independent of s and all choices of coordinates.

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Theorem

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Further, $e(\mathcal{E}, \tau)$ is often independent of τ, producing an enriched count of the given enumerative problem in $GW(k)$.

Non-orientable enumerative problems

Example (Lines and planes in $\mathbb{P}^4$)

(K.–Taylor '20) There are $\mathbf{3}\langle\mathbf{1}\rangle + \mathbf{2}\langle-\mathbf{1}\rangle$ lines meeting 6 general 2-planes in $\mathbb{P}^4$.

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Example (Lines and planes in $\mathbb{P}^4$)

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Further:

Conjecture (K.–Taylor '20)

For n even, there are

$$\frac{\mathbf{c(n-1)+i(n)}}{\mathbf{2}}\langle\mathbf{1}\rangle + \frac{\mathbf{c(n-1)-i(n)}}{\mathbf{2}}\langle\mathbf{-1}\rangle$$

lines meeting $2n-2$ codimension 2 hyperplanes in $\mathbb{P}^n$.

Thank you!