Categorifying quadratic zeta functions

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Algebra & Number Theory Seminar

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EMORY UNIVERSITY

Joint work with Jon Aycock

Introduction

Based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

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and an upcoming preprint, tentatively titled “Categorifying quadratic zeta functions” (with Jon Aycock).

Introduction

Mysterious setup question: you probably know the definition of *the* zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(and you may know some examples of other zeta functions), but what is *a* zeta function?

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The Riemann Zeta Function

The **Riemann zeta function** is a meromorphic function $\zeta_{\mathbb{Q}}(s)$ on the complex plane defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It has:

- a product formula $\zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1 - p^{-s}}$
- a functional equation $\xi(s) = \xi(1-s)$ for the "completed zeta function" $\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta_{\mathbb{Q}}(s)$
- information about the distribution of the primes
- a Riemann hypothesis that predicts all "nontrivial" zeroes have $\mathrm{Re}(s) = \frac{1}{2}$.

The Riemann Zeta Function

More generally, a **Dirichlet series** is a complex function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We will focus on the formal properties of Dirichlet series.

The coefficients $f(n)$ assemble into an **arithmetic function** $f : \mathbb{N} \to \mathbb{C}$. (Think: F is a generating function for f.)

Then $\zeta_{\mathbb{Q}}(s)$ is the Dirichlet series for $\zeta : n \mapsto 1$.

Arithmetic Functions

The space of arithmetic functions $A = \{f : \mathbb{N} \to \mathbb{C}\}$ form an algebra under convolution:

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

This identifies the algebra of formal Dirichlet series with A:

$$\begin{aligned} A &\longleftrightarrow DS(\mathbb{Q}) \\ f &\longmapsto F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\ f * g &\longmapsto F(s)G(s) \\ \zeta &\longmapsto \zeta_{\mathbb{Q}}(s) \\ ?? &\longmapsto \zeta_{\mathbb{Q}}(s)^{-1} \end{aligned}$$

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Number Fields

For a number field $K/\mathbb{Q}$, there is a zeta function $\zeta_K(s)$ defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}}{n^s}$$

where $I_K^+ = \{\text{ideals in } \mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

Like the Riemann zeta function, $\zeta_K(s)$ has:

- a product formula $\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}$
- a functional equation
- information about the distribution of prime ideals
- a Riemann hypothesis

Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits a map to $DS(\mathbb{Q})$:

$$\begin{aligned} N_* : A_K &\longrightarrow A \cong DS(\mathbb{Q}) \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_*\zeta \leftrightarrow \zeta_K(s) \\ \zeta^{-1} &\longmapsto ?? \end{aligned}$$

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Varieties over Finite Fields

Let X be an algebraic variety over $\mathbb{F}_q$. Its point-counting zeta function is the power series

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

which has:

- a product formula $Z(X,t) = \displaystyle\prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}$
- a functional equation
- an expression as a *rational function*
- a Riemann hypothesis which is a theorem!

Varieties over Finite Fields

Once again, we can formalize certain properties of $Z(X, t)$ in an algebra of arithmetic functions.

Let $Z_0^{\text{eff}}(X)$ be the set of effective 0-cycles on X, i.e. formal $\mathbb{N}_0$-linear combinations of closed points of X, written $\alpha = \sum m_x x$. We say $\beta \leq \alpha$ if $\beta = \sum n_x x$ with $n_x \leq m_x$ for all $x \in |X|$.

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta)g(\alpha - \beta).$$

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This time, there's no map to $DS(\mathbb{Q})$... but there's a map to the algebra of formal power series:

$$A_X \longrightarrow A_{\operatorname{Spec} \mathbb{F}_q} \cong \mathbb{C}[[t]]$$

$$f \leftrightarrow \sum_{n=0}^{\infty} f(n) t^n$$

$$f \longmapsto \text{“}\deg_*(f)\text{”}$$

$$\zeta \longmapsto \text{“}\deg_*(\zeta)\text{”} \leftrightarrow Z(X, t)$$

$$\zeta^{-1} \longmapsto ??$$

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What’s really going on?

What's really going on?

A, A_K and A_X are examples of the **reduced incidence algebra** of a poset.

Incidence Algebra of a Poset

Let $(\mathcal{P}, \leq)$ be a poset and define $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$. Call $\mathcal{P}$ **locally finite** if every interval is finite.

Definition

The **incidence coalgebra** of a locally finite poset $\mathcal{P}$ is the free k-vector space $C(\mathcal{P})$ on the set of intervals in $\mathcal{P}$, with comultiplication

$$[x, y] \longmapsto \sum_{z \in [x,y]} [x, z] \otimes [z, y].$$

The **incidence algebra** of $\mathcal{P}$ is the dual $I(\mathcal{P}) = \mathrm{Hom}(C(\mathcal{P}), k)$ with multiplication

$$f \otimes g \longmapsto (f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z])g([z, y]).$$

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Think: elements in $I(\mathcal{P})$ are like arithmetic functions on the intervals in $\mathcal{P}$.

Reduced Incidence Algebra of a Poset

Definition

The **reduced incidence algebra** of $\mathcal{P}$ is the subalgebra $\widetilde{I}(\mathcal{P}) \subseteq I(\mathcal{P})$ of functions that are constant on isomorphism classes of intervals.

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Example

For the division poset $(\mathbb{N}, |)$, every interval is isomorphic to $[1, n]$ for some n. For $f \in \widetilde{I}(\mathbb{N}, |)$, write $f(n) := f([1, n])$. Then

$$\widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q})$$

$$f \longmapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Fact: the zeta function $\zeta : [x, y] \mapsto 1$ always lives in $\widetilde{I}(\mathcal{P})$.

Numerical Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftrightarrows S_1 \leftrightarrows S_2 \cdots .$$

Example

A poset $\mathcal{P}$ determines a simplicial set $N\mathcal{P}$ with:

- 0-simplices $=$ elements $x \in \mathcal{P}$
- 1-simplices $=$ intervals $[x, y]$
- 2-simplices $=$ decompositions $[x, y] = [x, z] \cup [z, y]$
- etc.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \; \substack{\leftarrow\\ \to\\ \leftarrow\\ \to\\ \leftarrow} \; S_2 \cdots .$$

Example

More generally, any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

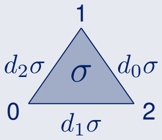
Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence coalgebra** of a decomposition set S is the free k-vector space $C(S) = \bigoplus_{x \in S_1} kx$ with comultiplication

$$\begin{aligned} C(S) &\longrightarrow C(S) \otimes C(S) \\ x &\longmapsto \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma. \end{aligned}$$



Numerical Incidence Algebras

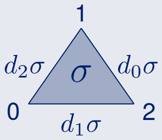
A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication

$$I(S) \otimes I(S) \longrightarrow I(S)$$

$$f \otimes g \longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma) g(d_0\sigma).$$



Numerical Incidence Algebras

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Key takeaways:

(1) A zeta function is $\zeta \in I(S)$ for some decomposition set S.

(2) Familiar zeta functions like $\zeta_K(s)$ and $Z(X,t)$ are constructed from some $\zeta \in \widetilde{I}(S)$ by pushing forward to another reduced incidence algebra which can be interpreted in terms of generating functions:

$$\text{e.g.} \quad \widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q}), \qquad \text{e.g.} \quad \widetilde{I}(\mathbb{N}_0, \leq) \cong k[[t]].$$

(3) Some properties of zeta functions can be proven in the incidence algebra directly:

$$\text{e.g.} \quad \zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1-p^{-s}} \longleftrightarrow \widetilde{I}(\mathbb{N}, |) \cong \bigotimes_p \widetilde{I}(\{p^k\}, |).$$

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** (“linear algebra with sets”):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

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$\longleftarrow$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

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So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(S_1 \xleftarrow{\;s\;} M \xrightarrow{\;t\;} * \right)$$

Abstract Incidence Algebras

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$$f = \left(\begin{array}{ccc} & M & \\ {}^{s}\swarrow & & \searrow^{t} \\ S_1 & & * \end{array} \right)$$

Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{array}{ccc} & S_1 & \\ {}^{\mathrm{id}}\swarrow & & \searrow \\ S_1 & & * \end{array} \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \left(\begin{array}{ccc} & M & \\ \overset{s}{\swarrow} & & \searrow \\ S_1 & & * \end{array} \right) \quad \text{and} \quad g = \left(\begin{array}{ccc} & N & \\ \overset{t}{\swarrow} & & \searrow \\ S_1 & & * \end{array} \right)$$

the convolution $f * g \in I(S)$ is represented by

$$f * g) = \left(\begin{array}{ccccccc} & & & P & & & \\ & & \swarrow & & \searrow & & \\ & S_2 & & & & M \times N & \\ \overset{d_1}{\swarrow} & & \overset{(d_2, d_0)}{\searrow} & & \underset{s \times t}{\swarrow} & & \searrow \\ S_1 & & & S_1 \times S_1 & & & * \end{array} \right)$$

Abstract Incidence Algebras

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- General: most* zeta functions can be produced this way
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- Fun! you can try it yourself

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

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In Aycock-K., we lift this formula to an equivalence of linear functors in $\widetilde{I}(\mathbb{Q}) := \widetilde{I}(\mathbb{N}, |)$:

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+$$

where $N : (I_K^+, |) \to (\mathbb{N}, |)$ is the norm and $\chi^+, \chi^- \in I(\mathbb{Q})$.

In the numerical incidence algebra, this becomes

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (\chi^+ - \chi^-) = \zeta_{\mathbb{Q}} * \chi.$$

Sketch of Proof

$$\boxed{N_* \zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Let $S = (\mathbb{N}, |)$ and $T = (I_K^+, |)$, so that $N : T \to S$ induces

$$N_* : \widetilde{I}(T) \longrightarrow \widetilde{I}(S), \quad f \longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right).$$

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

$$N_*\zeta_K = \left(\begin{array}{ccc} & T_1 & \\ {}^{N}\swarrow & & \searrow \\ S_1 & & * \end{array} \right)$$

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$$\zeta_{\mathbb{Q}} * \chi^- = \left(\begin{array}{ccccc} & & P^- & & \\ & \swarrow^{\alpha^-} & & \searrow & \\ & S_2 & & S_1 \times S_1^- & \\ \swarrow^{d_1} & & \searrow^{(d_2, d_0)} \quad \swarrow^{id \times j^-} & & \searrow \\ S_1 & & S_1 \times S_1 & & * \end{array} \right)$$

for a certain “vector” $j^- : S_1^- \to S_1$ representing χ^-.

Sketch of Proof

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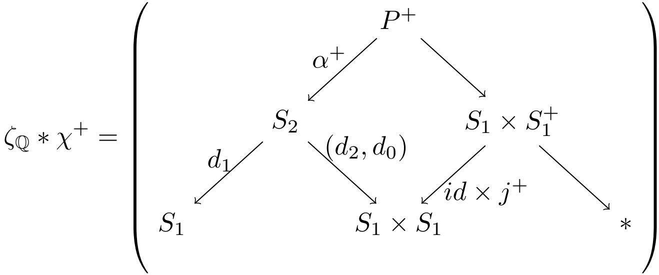

for a certain “vector” $j^+ : S_1^+ \to S_1$ representing χ^+.

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

So the formula is an equivalence of the following spans:

$$\left(S_1 \xleftarrow{N \sqcup d_1 \circ \alpha^-} T_1 \coprod P^- \longrightarrow * \right) \cong \left(S_1 \xleftarrow{d_1 \circ \alpha^+} P^+ \longrightarrow * \right)$$

These are shown to be equivalent prime-by-prime and then assembled into the global formula. □

Other Quadratic Zeta Functions: Euler Characteristic

For a ramified double cover $f : Y \to X$ of Riemann surfaces, the Riemann–Hurwitz formula says

$$\chi(Y) = 2\chi(X) + \sum_{y \in Y} (e_y - 1).$$

It should be possible to lift this to a formula in $I(SX)$, the incidence algebra of the simplicial complex of X, guided by the fact that

$$(1-t)^{-\chi(X)} = \sum_{n=0}^{\infty} \chi(X^{(n)}) t^n$$

is a generating function for the Euler characteristics of (symmetric powers of) X.

Other Quadratic Zeta Functions: Elliptic Curves

For an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

We are working on lifting this formula to the (reduced) incidence algebra $\widetilde{I}(Z_0^{\text{eff}}(\mathbb{P}^1))$. Pushing forward to $\widetilde{I}(Z_0^{\text{eff}}(\operatorname{Spec}\mathbb{F}_q)) \cong k[[t]]$, it already reads

$$(\pi_E)_*\zeta_E = (\pi_{\mathbb{P}^1})_*\zeta_{\mathbb{P}^1} * L(E).$$

More Dreams

These are some other things I want to do:

- Extend our approach to “higher degree covers”, e.g. for any abelian number field $K/\mathbb{Q}$, $\zeta_K(s) = \prod_{\chi} L(\chi, s)$
- Study the zeta function of an algebraic stack $\mathcal{X} \to X$ in terms of ζ_X, e.g. over $\mathbb{F}_q$, Behrend defines $Z(\mathcal{X}, t)$ for such a stack.
- Construct the right incidence algebra to house the motivic zeta function $Z_{mot}(X, t) = \sum_{n=0}^{\infty} [X^{(n)}] t^n$.
- Realize archimedean factors of completed zeta functions as elements of abstract incidence algebras, e.g. the factor at ∞ $\zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$.

Key insight: decomposition sets $\rightsquigarrow$ decomposition spaces

Thank you!