Wild Stacky Curves

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June 21, 2018

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Introduction

Goal:

Introduce wild ramification in stacky curves

Natural relation to moduli problems in characteristic $p > 0$

Moduli problems are ‘the right framework’ to study modular forms

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Natural relation to moduli problems in characteristic $p > 0$

Moduli problems are ‘the right framework’ to study modular forms
– at least, for an algebraic geometer like me :)

Motivation

Modular forms have a natural interpretation as sections of certain line bundles over a *moduli space* – hence the name ‘modular’ form.

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Here’s an example to keep in mind:

Example

$\Gamma = PSL_2(\mathbb{Z})$ acts on $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ by fractional linear transformations.

$Y = \mathfrak{h}/\Gamma$ is an affine curve, projective closure $X \cong \mathbb{P}^1_{\mathbb{C}}$ *is a Riemann surface*.

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$\mathbb{P}^1_{\mathbb{C}}$ = moduli space parametrizing isomorphism classes of elliptic curves (via j-invariant)

Moduli Problems

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- circles $\longleftrightarrow$ radius $+$ center
- circles up to isometry $\longleftrightarrow$ radius
- lines through the origin $\longleftrightarrow$ unit vector / $\pm$
- elliptic curves up to isomorphism $\longleftrightarrow$ j-invariant
- lines on a cubic surface $\longleftrightarrow$ there are exactly 27
- more examples?

Moduli Problems

More precisely, a moduli problem takes the form of a functor $P : \texttt{Schemes} \to \texttt{Sets}$.

Definition

P is **representable** if $P(X) = \mathrm{Hom}(X, M)$ for some scheme M – called a **moduli space** for P.

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- elliptic curves up to isomorphism $\longleftrightarrow \mathbb{A}^1_j$
- lines on a cubic surface $\longleftrightarrow$ zero-dim. variety with 27 points
- more examples?

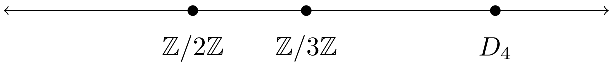
Moduli Problems via Stacks

Definition

A **stack** over a scheme X is:

- a family of categories $\mathcal{C} \to X$ (i.e. a fibered category)
- morphisms in each category $\mathcal{C}_x$ are all isomorphisms (i.e. $\mathcal{C}_x$ is a groupoid)

A nice class of stacks are Deligne-Mumford stacks. Think: “schemes with a finite automorphism group attached at each point”.



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Example

Y a scheme, G a group acting on Y: the *quotient stack*

$$\begin{array}{c} Y \\ \downarrow \\ [Y/G] \end{array}$$

parametrizes points of the quotient Y/G *along with their automorphisms*.

Moduli Problems via Stacks

Example (Important)

For every group G, there is a quotient stack

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that knows about all principal G-bundles:

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Example

Principal $GL_n(k)$-bundles $\leftrightarrow$ rank n vector bundles
In particular, principal $\mathbb{G}_m$-bundles $\leftrightarrow$ line bundles

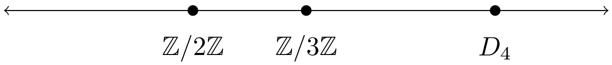
Stacky Curves

Stacky Curves

Definition

A **stacky curve** is a smooth, separated, connected, one-dimensional Deligne-Mumford stack $\mathcal{X}$ over a field k.

Think: smooth curve with some finite groups attached to a *finite number of points*



Call $\mathcal{X}$ **tame** if the orders of its stabilizers are prime to $\operatorname{char} k$. Otherwise, $\mathcal{X}$ is **wild**.

Stacky Curves

Definition

The **canonical ring** of $\mathcal{X}$ is the graded ring

$$R(\mathcal{X}) = \bigoplus_{k=0}^{\infty} H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k})$$

where $\omega_{\mathcal{X}}$ is the canonical sheaf on $\mathcal{X}$ (similar to that of a scheme).

When $\mathcal{X} = \mathfrak{h}/\Gamma_0(N)$ is a stacky modular curve,
$H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k}) \cong \mathcal{S}_{2k}(N)$, the space of weight $2k$ cusp forms of level N.

Stacky Curves

$$R(\mathcal{X}) = \bigoplus_{k=0}^{\infty} H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k})$$

Theorem (Voight & Zureick-Brown)

*There exists a combinatorial description of $R(\mathcal{X})$ when $\mathcal{X}$ is **tame**.*

Their proof uses: every **tame** stacky curve over an algebraically closed field is a root stack.

Explicit descriptions of spaces of modular forms in many cases

Stacky Curves

To extend this to **wild** stacky curves, we can use Artin-Schreier root stacks (work in progress).

This matters: dimension formulas for the spaces of modular forms for the congruence subgroups $\Gamma_0(p)$ and $\Gamma_1(p)\ldots$

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Recall: modular forms of weight $2k \leftrightarrow$ sections of a line bundle. Interesting question when $f(z) = g(z)^m$ for some $g(z)$ of lower weight.

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A **root stack** is a stacky version of X on which things like $L^{1/r}$ live.

Root Stacks

Rigorously:

Definition (Cadman, '07)

For a stack X, a line bundle $L \to X$ with section s and $r \geq 1$, the **rth root stack** $\sqrt[r]{(L,s)/X}$ is the fibre product

$$\begin{array}{ccccc} \sqrt[r]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & & x \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & & x^r \end{array}$$

Artin-Schreier Root Stacks

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Definition (K.)

For a stack X, a line bundle $L \to X$ and sections $s : X \to L$ and $f : X \to L^{\otimes m}$, the **Artin-Schreier root stack** with jump m is the fibre product

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(m,1)/\mathbb{G}_a] \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{P}(m,1)/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [x,y] \\ \downarrow \\ [x^p - xy^{m(p-1)}, y^p] \end{array}$$

Artin-Schreier Root Stacks

Example

Quotient of an AS curve by $\mathbb{Z}/p\mathbb{Z}$-action modeled by AS root stack:

$$[Y/(\mathbb{Z}/p\mathbb{Z})] \cong \wp_m^{-1}((\mathcal{O}_{\mathbb{A}^1}, x, f)/\mathbb{A}^1)$$

where $Y = \operatorname{Spec} k[x, y]/(y^p - y - f(x))$.

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Work in progress:

- Describe canonical ring for wild stacky curves
- Generalize to $\mathbb{Z}/p^a\mathbb{Z}$-covers
- Compute spaces of modular forms in char. p
- And more!