Understanding Artin-Schreier Covers ~~w/~~ Stacks

Goal:
- Describe AS covers of curves using stacks
- Natural applications to moduli spaces in char. $p > 0$
  - e.g. want to study modular forms for $\Gamma_0(p)$, $\Gamma_1(p)$
  - idea: $X_0(N)$, $X_1(N)$ arise as "stacky curves" with ramification data.

Outline:
- I. Moduli problems
- II. Kummer theory
- III. Artin-Schreier theory
- ~~IV. Stacky curves~~

I. Moduli Problems — "find an object that parametrizes ______" e.g. $\mathbb{P}^1$

Let X be a space and G a group. What properties of X are preserved up to G-action?

Define $\mathrm{Bun}_G(X)$ = category of principal G-bundles $P \to X$ / isomorphism.

Theorem: $\mathrm{Bun}_G(-)$ is representable, i.e. there exists a space BG and a natural isomorphism $\mathrm{Bun}_G(X) \cong [X, BG]$ for all X.

(BG = classifying space for G-bundles.)

Spaces $\mathrm{Bun}_G(X) \cong [X, BG]$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

universal G-bundle

up to iso.

up to htpy

Stacks $\{G\text{-bundles}\} \cong \mathrm{Hom}_{\text{stacks}}(X, BG)$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r] & BG = [\bullet/G] \end{tikzcd}$$

universal G-bundle of stacks

on the nose

Ex: GL_n-bundles $\longleftrightarrow$ rank n vector bundles

($\mathbb{G}_m$-bundles $\longleftrightarrow$ line bundles)

Prop: There is a bijection $(L,s) \longleftrightarrow [\mathbb{A}^1/\mathbb{G}_m](X) = \mathrm{Hom}(X,[\mathbb{A}^1/\mathbb{G}_m])$

(line bundle, section) (quotient stack)

n.b. $[\mathbb{A}^1/\mathbb{G}_m]$ may be thought of as an "infinitesimal thickening" of $B\mathbb{G}_m = [\bullet/\mathbb{G}_m]$.

II. Kummer Theory — way to understand cyclic G-covers when $p \nmid |G|$.

Natural question: for a line bundle $L \longrightarrow X$, X a scheme, and $r \geq 1$, does there exist another line bundle $E \longrightarrow X$ such that $E^{\otimes r} = L$?

Answer: Kummer theory (on stacks!)

Recall that a Kummer extension of fields is of the form L/k where

$$L = k[x]/(x^r - s) \quad \text{for some } r \geq 1,\ s \in k^\times.$$

It has $\mathrm{Gal}(L/k) \cong \mathbb{Z}/r\mathbb{Z}$.

To understand cyclic extensions in the language of stacks, we have the following construction of Cadman:

Def: The rth universal Kummer stack is the cover

$$\begin{tikzcd}
 & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[r, "x \mapsto x^r"] \arrow[d, phantom, "\cup"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, phantom, "\cup"] & \\
\text{(recall:} & {[\bullet/\mathbb{G}_m]} \arrow[r] & {[\bullet/\mathbb{G}_m]} & \text{)}
\end{tikzcd}$$

This allows us to construct rth roots of line bundles.

Def: For a scheme X, a line bundle $L \to X$ and a section $s : X \to L$, the <u>rth root stack</u> ${}^r\sqrt{(L,s)/X}$ is the fibre product

$$\begin{tikzcd}
{}^r\sqrt{(L,s)/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] & x \arrow[d, mapsto] \\
X \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} & x^r
\end{tikzcd}$$

induced by some "$(L^{1/r}, s^{1/r})$"

induced by (L,s)

Concretely, if $Y \to X$ is a $G = \mathbb{Z}/r\mathbb{Z}$ - Galois cover of curves, then

$$[Y/G] \cong {}^r\sqrt{(\mathcal{O}_X, s)/X}$$ where s encodes the branch data of the cover.

III. Artin-Schreier Theory

What to do when $\operatorname{char} k = p > 0$ and $p \mid |G|$?

Can we still find $E^{\otimes p} = L$?

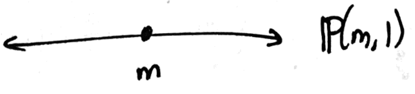
Recall than an Artin-Schreier extension of fields is of the form L/k where

$$L = k[x]/(y^p - y - a), \quad a \in k.$$

AS extensions are uniquely determined by their <u>ramification jump</u>, an integer $m \notin p\mathbb{Z}$ defined using the higher ramification filtration of $\mathrm{Gal}(L/k)$.

To understand AS covers using stacks, we need more data than just $[\mathbb{A}^1/\mathbb{G}_m]$...

For $m \in \mathbb{Z}$, the <u>weighted projective line</u> $\mathbb{P}(m,1)$ is $\mathbb{P}^1$ with one stacky point at 0 of order m:



Def: The <u>universal Artin–Schreier stack</u> w/ jump m is the cover of stacks

$$\begin{aligned} \wp_m : [\mathbb{P}(m,1)/\mathbb{G}_a] &\longrightarrow [\mathbb{P}(m,1)/\mathbb{G}_a] \\ [x,y] &\longmapsto [x^p - xy^{m(p-1)}, y^p] \end{aligned}$$

where $\mathbb{G}_a \curvearrowright \mathbb{P}(m,1)$ acts by $\lambda \cdot x = x + \lambda$ away from the stacky pt.

To study line bundles on wildly ramified curves, we have the following:

Prop: There is a bijection $(L, s, f) \longleftrightarrow \mathrm{Hom}(X, [\mathbb{P}(m,1)/\mathbb{G}_a])$.

(line bundle, section, section of $L^{\otimes m}$)

Explicitly, such a map $X \longrightarrow [\mathbb{P}(m,1)/\mathbb{G}_a]$ is $x \longmapsto [f(x), s(x)]$.

Def: For a scheme X, line bundle $L \to X$ and sections $s: X \to L$ and $f: X \to L^{\otimes m}$, the <u>Artin–Schreier root stack</u> of jump m is the fibre product

$$\begin{tikzcd}
\mathcal{X} = \wp_m^{-1}((L,s,f)/X) \arrow[r] \arrow[d] & {[\mathbb{P}(m,1)/\mathbb{G}_a]} \arrow[d, "\wp_m"] \\
X \arrow[r] & {[\mathbb{P}(m,1)/\mathbb{G}_a]}
\end{tikzcd}$$

determined by (L,s,f)

Here, the top row is determined by some (M, t, g) where $t: \mathcal{X} \to M$, $g: \mathcal{X} \to M^{\otimes m}$,

$$\begin{aligned} M^{\otimes p} &\xrightarrow{\sim} L \\ t^p &\longleftrightarrow s \\ g^p - t^{m(p-1)} g &\longleftrightarrow f. \end{aligned}$$

Ex: $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$

$Y = \operatorname{Spec} k[x,y]/(y^p - y - f(x))$ for $f \in k[x]$ deg. m, $p \nmid m$

Then $Y \longrightarrow \mathbb{A}^1$ has Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$ $\longleftrightarrow$ AS extension of function fields with jump m.

Goal: construct $[Y/G]$ as an AS root stack.

Sketch of proof: it's enough to show

$$\operatorname{Hom}(T, \mathcal{P}_m^{-1}((\mathcal{O}_{\mathbb{A}^1}, x, f)/\mathbb{A}^1)) \cong \operatorname{Hom}(T, [Y/G])$$

for any test scheme T.

On the left, $T \longrightarrow \mathcal{P}_m^{-1}((\mathcal{O}_{\mathbb{A}^1}, x, f)/\mathbb{A}^1)$ corresponds to

$$(\varphi : T \to \mathbb{A}^1,\ M \to T,\ t \in \Gamma(T, M),\ g : \Gamma(T, M^{\otimes m}),\ \psi)$$

where
$$\psi : M^{\otimes p} \xrightarrow{\sim} \varphi^* \mathcal{O}_{\mathbb{A}^1} = \mathcal{O}_T$$
$$t^p \longmapsto \varphi^* x$$
$$g^p - t^{m(p-1)} g \longmapsto \varphi^* f.$$

On the right, $T \longrightarrow [Y/G]$ corresponds to

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \\ T & \end{tikzcd}$$

where $P \to T$ is a G-bundle and $P \to Y$ is G-equivariant.

Send $(\varphi, M, t, g, \psi) \rightsquigarrow$

$$\begin{tikzcd} P = \{(g(z), t(z), z) \in M^{\otimes m} \times_T M \times T\} \arrow[d] \\ T. \end{tikzcd}$$

In general, the hope is that all covers of curves with wild ramification can be encoded with stacks in this way (work in progress):

- $\mathbb{Z}/p$ is described above, but need to understand general case by patching together affine AS curves
- $\mathbb{Z}/p^k\mathbb{Z}$ — Artin-Schreier-Witt theory?
- interface with (Kummer) root stacks.