Group members:

Warm-up: on your own, write down three sequences and at least the first 4 or 5 terms of each. Then show your group members the terms in one of your sequences and challenge them to find the general formula.

Problem 1. For each sequence below, find a function a_n that generates the sequence. Assume the starting index is $n = 1$. Do either of these sequences converge?

(a) $\left(\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \dots\right)$

(b) $\left(\frac{1}{2}, \frac{-1}{8}, \frac{1}{32}, \frac{-1}{128}, \dots\right)$

At the end of the packet, you will find a rigorous definition of the limit of a sequence. You are not required to learn this definition for quizzes or exams, but I wanted you to see what limits really look like "under the hood".

Problem 2. Determine whether the following sequences converge or diverge. For any that converge, find the value to which they converge.

(a) $a_n = \dfrac{1 - n - n^3}{6n^3 + 4}$

(b) $a_n = \sin\left(\dfrac{\pi}{2} + \arctan(n)\right)$

(c) $a_n = n - 10\sqrt{n}$

(d) $a_n = 2b_n - \dfrac{c_n}{b_n}$, where the sequence b_n converges to $\frac{1}{2}$ and the sequence c_n converges to 3.

Not all sequences come from nice real-valued functions, i.e. $a_n = f(n)$ for some function $f(x)$ (see 3.2.4 in the textbook). Here's an example.

Problem 3. Consider the sequence defined by $a_n = \dfrac{2^n}{n!}$.

(a) Find an N such that $a_n < \frac{1}{10}$ whenever $n > N$.

(b) Find an N such that $a_n < \frac{1}{100}$ whenever $n > N$.

(c) Make a guess for the value to which a_n converges.

Problem 4. Here are some more examples to practice with. For each sequence, compute its limit or explain why the limit does not exist.

(a) $a_n = \dfrac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}$

(b) $b_n = \dfrac{\sin(n)}{n}$

(c) $c_n = e^{-n^2}$

(d) $e_n = \left(1 + \dfrac{1}{n}\right)^n$

Problem 5. Make an educated guess at what the sequence $y_n = \sqrt{n^2 + n} - n$ converges to. Then verify your guess.

Problem 6. Let $F_1 = 1, F_2 = 1$ and define the rest of the sequence recursively by

$$F_n = F_{n-2} + F_{n-1}.$$

The sequence (F_n) is called the Fibonacci sequence; its terms are called *Fibonacci numbers*.

(a) Compute F_3, F_4, F_5, F_6. Can you think of a function $F(x)$ such that $F(n) = F_n$ for all $n \in \mathbb{N}$?

(b) Next, consider the sequence (a_n) defined by the ratios of consecutive Fibonacci numbers:

$$a_n = \frac{F_{n+1}}{F_n}.$$

Assuming that (a_n) converges (this is slightly tricky to show), prove that the limit of the sequence is the *golden ratio* φ, defined as the unique positive real number satisfying

$$\varphi^2 - \varphi - 1 = 0.$$

(Note: using the quadratic formula, we can see that the positive root of $x^2 - x - 1 = 0$ is $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$, which is another way to define φ.)

Here are some useful limit properties that parallel the properties for limits of a real-valued function we learned in calculus I.

Theorem. Let (a_n) and (b_n) be sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then:

(i) For all real numbers k, the sequence (ka_n) converges to ka.

(ii) The sequence $(a_n + b_n)$ converges to $a + b$. Similarly, $(a_n - b_n)$ converges to $a - b$.

(iii) The sequence $(a_n b_n)$ converges to ab.

(iv) Assume $a_n \neq 0$ for all n and $a \neq 0$. Then $\left(\frac{1}{a_n}\right)$ converges to $\frac{1}{a}$.

(v) If $a_n > 0$ for all n, then for all $k > 0$, the sequence (a_n^k) converges to a^k.

(vi) If $a_n > 0$ for all n, then (a_n) diverges to $+\infty$ if and only if $\left(\frac{1}{a_n}\right)$ converges to 0.

As an exercise, see if you can prove each of these using the ϵ–N definition of the limit of a sequence on the next page.

A sequence (a_n) **converges** to a real number L, written $\lim_{n\to\infty} a_n = L$ or simply $(a_n) \to L$, if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|a_n - L| < \epsilon.$$

If this holds, we say L is the **limit of the sequence** (a_n).

In English: a sequence converges to a target value if we can make the terms of the sequence as close as we want to the target by taking terms far enough along in the sequence.

Problem 7. Make an educated guess at what the sequence $a_n = \frac{1}{n^2}$ converges to and then prove, using the above definition, that (a_n) converges to that value.