Group members:

Warm-up: with your group, discuss the differences between an infinite sequence and an infinite series. What does it mean for each to converge or diverge?

Problem 1. Consider the infinite series $\sum_{n=0}^{\infty}(-1)^{n+1}$. Decide whether this series converges or diverges by examining the associated sequence of partial sums.

Problem 2. Consider the infinite series $\sum_{n=0}^{\infty} \frac{1}{2^n}$. To figure out if this series converges or diverges, write out its Nth partial sum:

$$s_N = \sum_{n=0}^{N} \frac{1}{2^n} =$$

Multiply through by $\frac{1}{2}$:

$$\frac{1}{2} s_N =$$

Then subtract these two expressions and cancel out like terms:

$$\frac{1}{2} s_N = s_N - \frac{1}{2} s_N =$$

Now solve for s_N to get a formula for this Nth partial sum:

$$s_N =$$

To evaluate the infinite series, take the limit of your formula for the partial sum:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \lim_{N\to\infty} s_N =$$

A series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$ is called a *geometric series* with coefficient a and ratio r.

Theorem 1 (Geometric Series). A geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

WARNING: It is important to remember the difference between a sequence (a_n) and the series it defines, $\sum_{n=1}^{\infty} a_n$. Often, one can converge while the other diverges, or they can converge to different values.

Problem 3. State whether the following geometric series converge or diverge, and if they converge, compute their value.

(a) $\displaystyle\sum_{n=0}^{\infty} -5\left(\frac{2}{3}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{2^{n+2}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{2n}$

Problem 4. Determine whether the following series converge or diverge. If any converge, find their sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + 2^{2n}}{5^{n-1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$ Hint: try to find a formula for the partial sums.

The series in (b) is known as a "telescoping series" – can you see why? In general, telescoping series and geometric series are the only infinite series whose limits we can compute (if they converge). In general, it may be difficult or impossible to determine the exact value of a convergent series. We next explore some tools for answering the question "does this series converge or diverge?" without necessarily knowing the value.

Theorem 2 (Divergence Test). If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if the sequence (a_n) does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 3 (Ratio Test). Let $\sum_{n=1}^{\infty} a_n$ be a series and set $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then

(a) If L exists and $L < 1$, then the series converges absolutely.

(b) If L exists and $L > 1$, or if $L = \infty$, then the series diverges.

(c) If $L = 1$, then **the test is inconclusive**.

Problem 5. For each of the following, determine if the series converges or diverges, stating any appropriate tests for convergence. If possible, compute what the series converges to.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n-2n^2}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(n+1)}$ (Hint: try to rewrite the fraction as a difference of fractions...)

(c) $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^{n+1}2^{2n}}{3^n}$

(d) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{n} - \frac{1}{n-2}\right)$

(e) $\displaystyle\sum_{n=0}^{\infty} \left[\sec\left(\frac{\pi}{4}\right)\right]^n$

Here are two more tests that we didn't learn about in the video.

Theorem 4 (Integral Test). Let $f(x)$ be a continuous function which is nonnegative and decreasing on $[1, \infty)$ and consider the sequence $a_n = f(n)$ for $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges if and only if} \quad \int_1^{\infty} f(x)\, dx \quad \text{converges.}$$

Theorem 5 (p-Series Test). The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Problem 7. Use the Integral Test to prove the p-Series Test. You may use the p-test for improper integrals in your proof.

Problem 8. Does the integral test apply to $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$? Explain why or why not.

Problem 9. For what value of c does $\displaystyle\sum_{n=0}^{\infty} \mathrm{e}^{cn} = 10$?

Problem 10. Here are a few more series to practice with, using all of the tests you've learned. For each, decide if the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{24}{\sqrt[8]{n^5}}$

(b) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\mathrm{e}^{n/2}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\mathrm{e}^n}{n}$