Group members:

Warm-up: find the formula for a joint probability distribution representing the probability of choosing a random point (X, Y) uniformly in the unit circle $C = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Problem 1. Let X and Y be two continuous random variables with joint distribution

$$f(x,y) = \begin{cases} x + ky^2, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the constant k. Then compute $P\left(0 \leq X \leq \frac{1}{2} \text{ and } 0 \leq Y \leq \frac{1}{2}\right)$.

Problem 2. Let X and Y be two continuous random variables with joint distribution

$$f(x,y) = \begin{cases} kx^2y, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k.

(b) Find the marginal distributions f_X and f_Y.

(c) Compute the probability that X is at least $\frac{1}{2}$.

(d) Compute the probability that X is at least twice as large as Y.

Problem 3. Let X and Y be two continuous random variables with joint distribution $f(x,y) = \frac{2}{3}(x+y)^3$ on the unit square with vertices $(0,0),(1,0),(0,1)$ and $(1,1)$. Compute the probability that X and Y sum to more than 1.

Problem 4. Let X and Y be two continuous random variables with joint distribution

$$f(x,y) = \begin{cases} 6e^{-(2x+3y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?

(b) Compute $P(X > Y)$.

Problem 5. Let X and Y be two continuous random variables with joint distribution $f(x, y) = x + y$ on the unit square again. Find the following expected values:

(a) $E[X]$

(b) $E[Y^2]$

(c) $E[XY^2]$

Problem 6. Let X and Y be two continuous random variables with joint distribution

$$f(x,y) = \begin{cases} 6xy, & 0 \le x \le 1, 0 \le y \le \sqrt{x} \\ 0, & \text{otherwise.} \end{cases}$$

Find the covariance $\operatorname{cov}(X, Y)$.