

# A Modular Approach to Supersingular Mass Formulas

Andrew Kobin

andrewkobin.com

Emory University

ajkobin@emory.edu

some joint work in progress with:

- Santiago Arango-Piñeros & Sun Woo Park
- Eran Assaf

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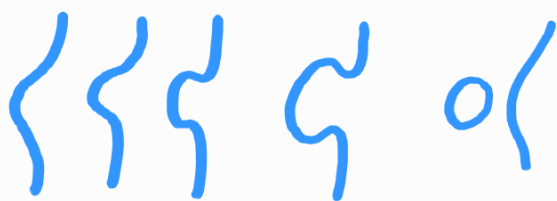
## Introduction

Key object of interest:

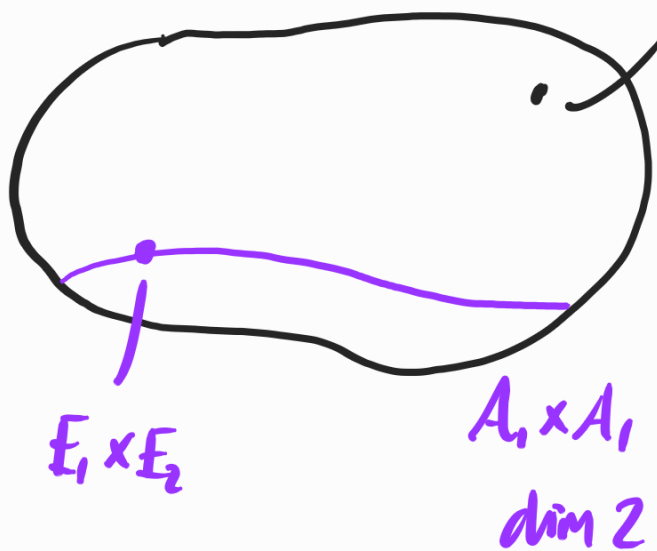
(PPAVs)

$A_g$  = moduli space of principally polarized

abelian varieties of dim.  $g$ ,  
up to isomorphism /  $K$



$$A_1 = M_{1,1} \cong \mathbb{A}_K^1$$



$J(C)$ ,  $C$  genus 2

$A_2$  dim 3

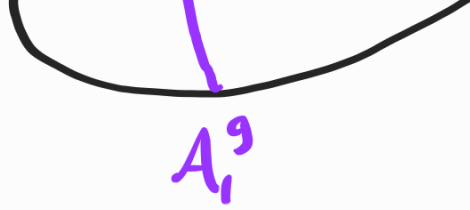
rational 3-fold

$A_{g-1} \times A_1$



$J(C)$ ,  $C$  genus 3

$A_g$ ,  $g \geq 3$



$$\dim \frac{g(g+1)}{2}$$

general type for  $g \geq 7$

## Compactifications:

- $A_1 = \mathcal{M}_{1,1}$  has a standard compactification  $\bar{\mathcal{M}}_{1,1}$  by adding nodal cubics.
- For  $g \geq 2$ ,  $A_g$  admits many compactifications, e.g. Satake, Chai-Faltings, Baily-Borel.

Our setting:  $A_g/K$  for  $K = \bar{\mathbb{F}}_p$  (char.  $p$ ).

New phenomena in characteristic  $p$  vs.

characteristic 0: supersingular AVs.

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## Supersingular Abelian Varieties

For  $g=1$ , an elliptic curve  $E/\mathbb{F}_q$  has a  $p$ -torsion subgroup  $E[p]$  with

$$E[p](K) \subseteq \mathbb{Z}/p\mathbb{Z}.$$

$E$  is supersingular if  $\underline{E[p]}(K) = \{0\}$ .

(otherwise ordinary)

$$\text{rk } E = v$$

$$\text{if } E[p] \cong \mathbb{Z}/p^v\mathbb{Z}$$

Many equivalent conditions:

•  $E$  supersingular  $\iff$   $p$ -rank 0

•  $\text{End}(E)$  is an order in a quaternion algebra



- $\text{Lnd}(L)$  is an odd in a quadratic extension.
- $a_g(E) = 0$ , where  $L(E, t) = 1 - \underbrace{a_g(E)t + qt^2}_{\text{Hasse invariant}}$
- $\text{Tr}(\text{Frob}_q | H^1(E)) = 0$
- $NP(E[p^\infty])$  has slope  $1/2$

For higher dimensional abelian varieties, not all of these are equivalent.

An abelian variety  $A/\mathbb{F}_q$  of dim.  $g$  is **supersingular** if it is isogenous to a product  $E_1 \times \dots \times E_g$  of s.s. elliptic curves.

$A \rightarrow$

We say  $A$  is **superspecial** if it is isomorphic

to such a product. | FACT: if  $g \geq 2$ ,  
 $\Leftrightarrow A \cong E_0^g$ .

Notes:

•  $A$  s.s.  $\Rightarrow A[p](K) = \{0\}$

but  $(\Leftarrow)$  only holds for  $g = 1, 2$ .

•  $A$  s.sp.  $\Leftrightarrow a(A) := \dim \text{Hom}_k(\alpha_p, A) = 0$ .

• (Deuring) There exists a s.s. elliptic curve  $E/\mathbb{F}_p$  for all  $p$ .

$$\# M_{1,1}^{ss}(\mathbb{F}_q) = ?$$

Q: How many s.s. elliptic curves are there?

" " " PPAVs " " ?

$$\# A_g^{ss}(\mathbb{F}_q), \# A_g^{ssp}(\mathbb{F}_q) = ?$$

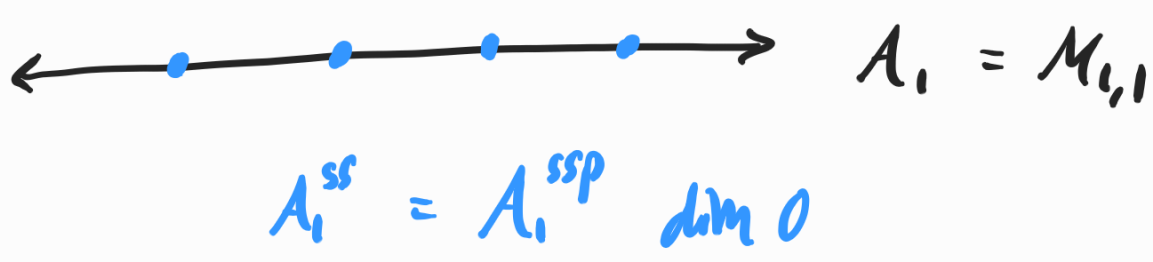
**Theorem (Ehler-Deuring-Tate)** There are

Theorem (Lichtenberg, Deuring, Igusa) There are

$$\left[ \frac{p-1}{12} \right] + \begin{cases} 0, & \text{if } p \equiv 1 \pmod{12} \\ 1, & \text{if } p \equiv 2, 3, 5, 7 \pmod{12} \\ 2, & \text{if } p \equiv 11 \pmod{12} \end{cases}$$

supersingular elliptic curves over  $K = \overline{\mathbb{F}}_p$ .

Moreover, the supersingular locus is defined over  $\mathbb{F}_{p^2}$ .



In dimension  $g$ ,  $\dim A_g$

- $\dim(Ag^{ss}) = \left\lfloor \frac{g}{4} \right\rfloor < \frac{g(g+1)}{2}$ .

- $\dim(Ag^{ssp}) = 0$ .

- point counts are known in some cases.

Nicer formulas come from masses, rather than point counts.

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## Mass Formulas

Theorem (Eichler 1938, Dewing 1941)

There are

$\frac{p-1}{24}$  supersingular elliptic curves over

$K = \overline{\mathbb{F}_p}$ , weighted by number of automorphisms.

That is,

$$\text{mass}(M_{1,1}^{ss}(K)) := \sum_{(E) \in M_{1,1}^{ss}(K)} \frac{1}{|\text{Aut}(E)|} = \frac{p-1}{24}$$

**Ex** For  $p = 2$ , there is a unique s.s.

elliptic curve

$$E: y^2 + y = x^3$$

with  $\text{Aut}(E) \cong \mathbb{Q} \rtimes \mathbb{Z}/3\mathbb{Z} \leftarrow \text{order } 24$

$\cap 1$

$\text{End}(E) \cong \mathbb{H}$  ← Hurwitz integers in Hurwitz quaternions

The usual proof exploits the equivalence

$$M_{1,1}^{ss}(K) \longleftrightarrow \left\{ \begin{array}{l} \text{maximal left } \mathcal{O}\text{-ideals} \\ L \subseteq \mathcal{O} = \text{End}(E_0) \end{array} \right\}$$

↖ fixed, s.s.

$$E \longmapsto \text{Hom}(E_0, E)$$

AV

$$\text{Aut}(E) \longleftrightarrow \text{Aut}_{\mathbb{H}}(L)$$

which preserves automorphisms.

$$\text{So } \text{mass}(M_{1,1}^{ss}(K)) = \sum_{E \in M_{1,1}^{ss}(K)} \frac{1}{|\text{Aut}(E)|}$$

$$= \sum_{L \subseteq \mathcal{O}} \frac{1}{|\text{Aut}(L)|} = \frac{p-1}{24}.$$

$$\#M_{1,1}^s(\overline{\mathbb{F}}_p) = h(\mathcal{O})$$

Shimura mass formula

## Modular proof (essentially Katz-Mazur):

$\bar{M}_{1,1}$  is a Deligne-Mumford stack over  $K$

$\simeq \mathbb{P}(4,6)$

in all characteristics.

$|G| < \infty$



The Hodge bundle  $\mathcal{L}$  is a (stacky) line

bundle on  $\bar{M}_{1,1}$  and its sections are

mod  $p$  modular forms (of level 1):

$$\bigoplus_{k=0}^{\infty} H^0(\bar{M}_{1,1}, \mathcal{L}^{\otimes k}) \cong \bigoplus_{k=0}^{\infty} M_k(\Gamma(1), \bar{\mathbb{F}}_p).$$

$SL_2(\mathbb{Z})$   
 $\downarrow$   
 $\uparrow$   
 $k$

The Hasse invariant  $a_p(-)$  is a modular

form of weight  $p-1$  with vanishing

locus  $M_{1,1}^{ss}$ , so

$$\deg(\mathcal{L}^{\otimes p-1}) = \deg(a_p) = \text{mass}(M_{1,1}^{ss}).$$

$\sum_{x \in V(a_p)} \frac{1}{|A_{\text{Aut}(x)}|}$

On the other hand, the cusp form  $\Delta$  of weight 12 has degree  $1/2$ , so

$$\deg(\mathcal{L}^{\otimes 12}) = \deg(\Delta) = \frac{1}{2}$$

and therefore  $\deg(\mathcal{L}^{\otimes p-1}) = \frac{p-1}{24}$ .  $\square$

(originally in Viehweg's *Quat. Alg's*)

**Theorem (Arango-Piñeros — K. — Park 2024+ε)**

Let  $N \geq 2$  be prime to  $p$ . Then the mass of supersingular elliptic curves with level  $N$  is



$$\text{mass}(\mathcal{Y}_0(N)^{ss}(K)) = \frac{p-1}{24} \cdot N \prod_{\substack{\ell|N \\ \text{prime}}} \left(1 + \frac{1}{\ell}\right).$$

↑  
stacky modular  
curve

=  $\deg(\mathcal{Y}_0(N) \xrightarrow{\hat{\pi}_N} \mathcal{M}_{1,1})$

$\pi_N^* \mathcal{L}, \pi_N^* \omega_p, \dots$

Recall that for  $N=1$ , the supersingular locus is defined over  $\mathbb{F}_p^2$ .

Then the minimal density of  $\mathcal{Y}_0(1)^{ss} = \mathcal{M}_{1,1}^{ss}$  is defined to be

$$\delta := \frac{\text{mass}(\mathcal{M}_{1,1}^{ss}(\mathbb{F}_p^2))}{\text{mass}(\mathcal{M}_{1,1}(\mathbb{F}_p^2))}.$$

as a DM stack

Then the Eichler–Deuring formula shows

$$\delta = \frac{p-1}{12p^2-14}.$$

"Supersingular elliptic curves over  $\mathbb{F}_p^2$   
are more sparse as  $p \rightarrow \infty$ ."

Adding in level structure, we can define

$$\delta_N := \frac{\text{mass}(\mathcal{Y}_0(N)^{ss}(K_N))}{\text{mass}(\mathcal{Y}_0(N)(K_N))}$$

where  $K_N \geq \mathbb{F}_p$  is the minimal field  
of definition of  $\mathcal{X}_0(N)^{ss}$ .

Corollary (AP-K-P 2024+ $\epsilon$ )

For  $N$  prime

to  $p$ ,

$$(p-1)N \prod (1 + \frac{1}{p})$$

$$S_N \geq \frac{(p-1)N \prod_{i=1}^N (1+i)}{24 \#y_0(N)(\mathbb{F}_p^{12\phi(N)})}$$

↑  
torient of  $N$

Work in progress:

- Compute mass  $(y_0(N)(\mathbb{F}_p^{12\phi(N)}))$  explicitly, at least for the genus 0  $y_0(N)$ .

This refines the lower bound for  $S_N$ .

- Compute asymptotic for  $S_N$  as  $p \rightarrow \infty$ .

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Mass Formulas in Higher Dimensions

Full formula

# The Eichler - Deuring Formula

$$\sum_{E \in \mathcal{M}_{g,n}^{ss}(K)} \frac{1}{|\text{Aut}(E)|} = \frac{p-1}{24}$$

has been generalized to the superspecial

locus  $A_g^{ssp} \subseteq A_g$  for  $g \geq 2$ :

Theorem (Hashimoto - Ibukiyama 1980  
Ekedahl 1987)

For any  $p$ ,  $g \geq 1$ ,

$$\text{mass}(A_g^{ssp}(\overline{\mathbb{F}}_p)) = \frac{(-1)^{g(g+1)/2}}{2^g} \prod_{j=1}^g g(1-2j) \prod_{k=1}^g (p^k + (-1)^k).$$

"deg<sub>p</sub>" =  $\frac{g(g+1)}{2}$

Bernoulli #'s

$$\boxed{\text{Ex}} \quad \text{mass} (A_2^{\text{ssp}}(\overline{\mathbb{F}}_p)) = \frac{(p-1)(p^2+1)}{5760} = 2^7 \cdot 3^2 \cdot 5$$

$$\boxed{\text{Ex}} \quad \text{mass} (A_3^{\text{ssp}}(\overline{\mathbb{F}}_p)) = \frac{(p-1)(p^2+1)(p^3-1)}{2903040} = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$$

For other strata in the supersingular locus,  
mass formulas have been found by Yu (2003),  
Yu-Yu (2009), Karemaker-TobuKo-Yu (2021).

Joint work in progress with E. Assaf: Recover

these mass formulas using stacky point

counts of vanishing loci of appropriate

mod  $p$  Siegel modular forms.

Theorem (Assaf - K. 2024 + ε) For  $g=2$ ,

there are mod  $p$  Siegel modular forms

$$Ha_1 \in H^0(A_2, \mathcal{L}^{\otimes p-1}) = M_{p-1}(\overline{\mathbb{F}}_p)$$

and  $Ha_2 \in H^0(A_2, \mathcal{L}^{\otimes p^2+1}) = M_{p^2+1}(\overline{\mathbb{F}}_p)$

(partial Hasse invariants)

such that  $\text{mass}(A_2^{ssp}(\overline{\mathbb{F}}_p))$  can be computed  
in terms of the stacky vanishing loci of  
 $Ha_1$  and  $Ha_2$ .

Future directions:

- Use partial Hasse invariants  $Ha_1, \dots, Ha_g$   
to compute the mass of  $A_g^{ssp}(\overline{\mathbb{F}}_p)$ .

to compute  $\Gamma$

- Use Siegel modular forms to compute the masses of other strata in  $Ag^{ss}$ .

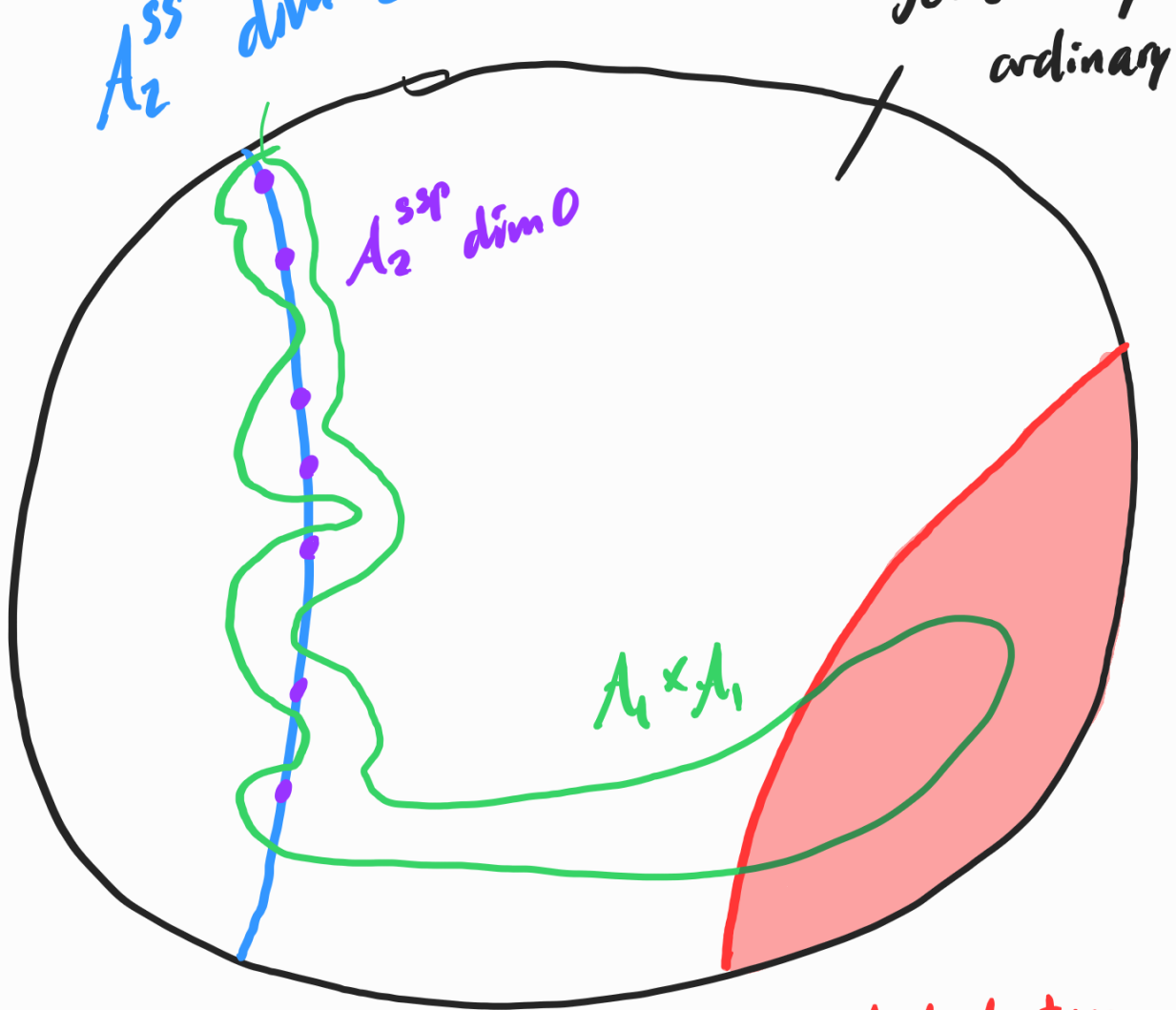
- Add level structure  $\rightsquigarrow$  asymptotics?

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THANK YOU!  
QUESTIONS?

$\dim 1 \leftrightarrow V(H_{2,2})$

generically



$$A_1 \times A_1 \cap A_2^{ss}$$

$$A_2 \quad \text{dim } 3$$



