A Modular Approach to Supersingular Mass Formulas

Andrew Kobin   Emory University

andrewkobin.com   ajkobin@emory.edu

some joint work in progress with:

- Santiago Arango-Piñeros & Sun Woo Park
- Eran Assaf

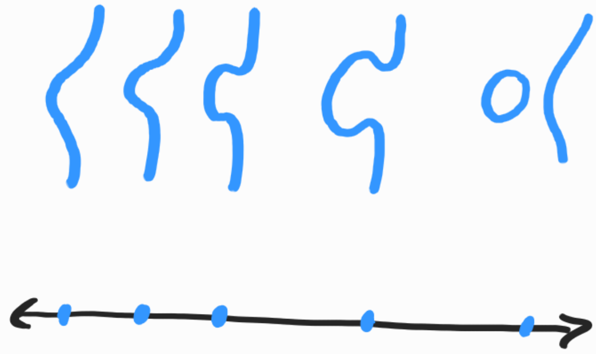
Introduction

Key object of interest:

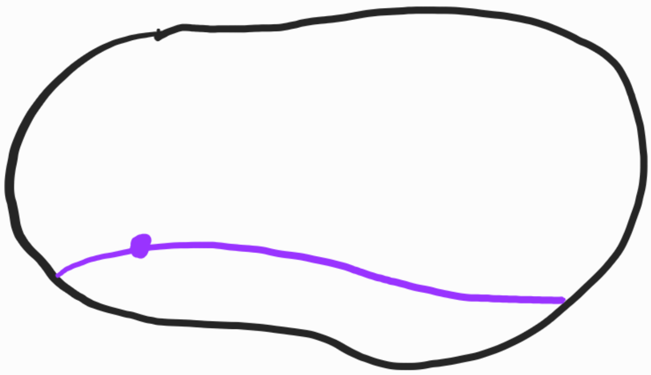
(PPAVs)

A_g = moduli space of principally polarized abelian varieties of dim. g,

up to isomorphism $/K$

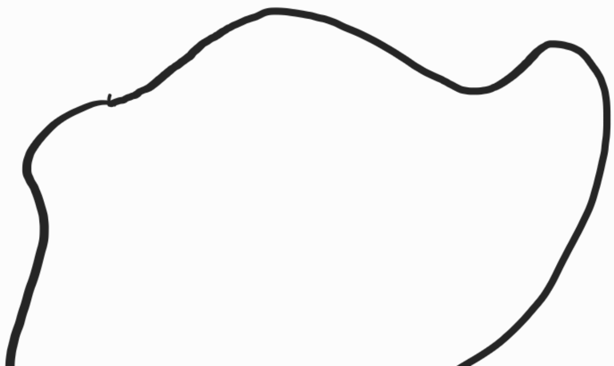


$A_1 = M_{1,1} \cong \mathbb{A}^1_K$



A_2 dim 3

rational 3-fold



A_g, $g \geq 3$

dim $\frac{g(g+1)}{2}$

general type for $g \geq 7$

Compactifications:

- $A_1 = M_{1,1}$ has a standard compactification $\overline{M}_{1,1}$ by adding nodal cubics.

- For $g \geq 2$, A_g admits many compactifications, e.g. Satake, Chai-Faltings, Baily-Borel.

Our setting: A_g / K for $K = \overline{\mathbb{F}_q}$ (char. p).

New phenomenon in characteristic p vs.

characteristic 0 : supersingular AVs.

Supersingular Abelian Varieties

For $g = 1$, an elliptic curve $E/\mathbb{F}_q$ has a p-torsion subgroup $E[p]$ with

$$E[p](k) \subseteq \mathbb{Z}/p\mathbb{Z}.$$

E is supersingular if $E[p](K) = \{0\}$.

(otherwise ordinary)

Many equivalent conditions:

- E supersingular
- $\mathrm{End}(E)$ is an order in a quaternion alg

- $\mathrm{End}(E)$ is an order in a quaternion alg.
- $a_q(E) = 0$, where $L(E,t) = 1 - a_q(E)t + qt^2$
- $\mathrm{Tr}(\mathrm{Frob}_q \mid H^1(E)) = 0$
- $NP(E[p^\infty])$ has slope $1/2$

For higher dimensional abelian varieties, not all of these are equivalent.

An abelian variety $A/\mathbb{F}_q$ of dim. g is **supersingular** if it is isogenous to a product $E_1 \times \cdots \times E_g$ of s.s. elliptic curves.

We say A is **superspecial** if it is isomorphic to such a product.

Notes:

- A s.s. $\implies A[p](K) = \{0\}$

  but ($\Leftarrow$) only holds for $g = 1, 2$.

- A s.sp. $\iff a(A) := \dim \mathrm{Hom}_k(\alpha_p, A) = 0$.

- (Deuring) There exists a s.s. elliptic curve $E/\overline{\mathbb{F}}_p$ for all p.

Q: How many s.s. elliptic curves are there?

" " " PPAVs " " ?

Theorem (Eichler-Deuring-Igusa) There are

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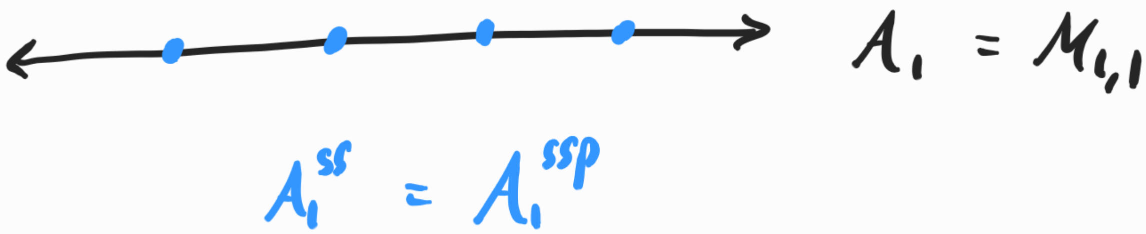
$$\left\lfloor \frac{p-1}{12} \right\rfloor + \begin{cases} 0, & \text{if } p \equiv 1 \pmod{12} \\ 1, & \text{if } p \equiv 2, 3, 5, 7 \pmod{12} \\ 2, & \text{if } p \equiv 11 \pmod{12} \end{cases}$$

supersingular elliptic curves over $K = \overline{\mathbb{F}_p}$.

Moreover, the supersingular locus is defined over $\mathbb{F}_{p^2}$.



In dimension g,

- $\dim(A_g^{ss}) = \left[\frac{g^2}{4}\right] < \frac{g(g+1)}{2}$.

- $\dim(A_g^{ssp}) = 0$.

- point counts are known in some cases.

Nicer formulas come from <u>masses</u>, rather than point counts.

Mass Formulas

Theorem (Eichler 1938, Deuring 1941) There are supersingular elliptic curves over

$K = \overline{\mathbb{F}_p}$, weighted by number of automorphisms.

That is,

$$\mathrm{mass}(\mathcal{M}_{1,1}^{ss}(K)) := \sum_{E \in \mathcal{M}_{1,1}^{ss}(K)} \frac{1}{|\mathrm{Aut}(E)|} =$$

Ex For $p = 2$, there is a unique s.s.

elliptic curve

$$E : y^2 + y = x^3$$

with $\mathrm{Aut}(E) \cong Q \rtimes \mathbb{Z}/3\mathbb{Z}$ ← order 24

$\cap|$

$$\mathrm{End}(E) \cong \mathbb{H}$$ Hurwitz integers in Hurwitz quaternions

The usual proof exploits the equivalence

$$\begin{aligned} \mathcal{M}_{1,1}^{ss}(K) &\longleftrightarrow \left\{ \begin{matrix} \text{maximal left } \mathcal{O}\text{-ideals} \\ L \subseteq \mathcal{O} = \mathrm{End}(E_0) \end{matrix} \right\} \\ E &\longmapsto \mathrm{Hom}(E_0, E) \end{aligned}$$

fixed, s.s.

which <u>preserves</u> <u>automorphisms</u>.

So $$\mathrm{mass}(\mathcal{M}_{1,1}^{ss}(K)) = \sum_{E \in \mathcal{M}_{1,1}^{ss}(K)} \frac{1}{|\mathrm{Aut}(E)|}$$

$$= \sum_{L \subseteq \mathcal{O}} \frac{1}{|\mathrm{Aut}(L)|} = \frac{p-1}{24}.$$

Shimura mass formula

Modular proof (essentially Katz–Mazur):

$\overline{\mathcal{M}}_{1,1}$ is a Deligne–Mumford stack over K in all characteristics.

The Hodge bundle $\mathcal{L}$ is a (stacky) line bundle on $\overline{\mathcal{M}}_{1,1}$ and its sections are mod p modular forms (of level 1):

$$\bigoplus_{k=0}^{\infty} H^0(\overline{\mathcal{M}}_{1,1}, \mathcal{L}^{\otimes k}) \cong \bigoplus_{k=0}^{\infty} M_k(\Gamma(1), \overline{\mathbb{F}}_p).$$

The Hasse invariant $a_p(-)$ is a modular form of weight $p-1$ with vanishing locus $\mathcal{M}_{1,1}^{ss}$, so

$$\deg(\mathcal{L}^{\otimes p-1}) = \deg(a_p) = \text{mass}(\mathcal{M}_{1,1}^{ss}).$$

On the other hand, the cusp form Δ of weight 12 has degree $1/2$, so

$$\deg(\mathcal{L}^{\otimes 12}) = \deg(\Delta) = \frac{1}{2}$$

and therefore $\deg(\mathcal{L}^{\otimes p-1}) = \dfrac{p-1}{24}$. $\square$

$\boxed{\text{Theorem (Arango-Piñeros — K. — Park 2024+}\varepsilon)}$

Let $N \geq 2$ be prime to p. Then the mass of supersingular elliptic curves with level N is

$$\mathrm{mass}\left(\mathcal{Y}_0(N)^{ss}(K)\right) = \frac{p-1}{24} \cdot N \prod_{\substack{\ell \mid N \\ \text{prime}}} \left(1 + \frac{1}{\ell}\right).$$

stacky modular curve

Recall that for $N = 1$, the supersingular locus is defined over $\mathbb{F}_{p^2}$.

Then the minimal density of $\mathcal{Y}_0(1)^{ss} = \mathcal{M}_{1,1}^{ss}$ is defined to be

$$\delta := \frac{\mathrm{mass}\left(\mathcal{M}_{1,1}^{ss}(\mathbb{F}_{p^2})\right)}{\mathrm{mass}\left(\mathcal{M}_{1,1}(\mathbb{F}_{p^2})\right)}.$$

Then the Eichler–Deuring formula shows

$$\delta = \frac{p-1}{12p^2 - 14}.$$

"Supersingular elliptic curves over $\mathbb{F}_{p^2}$ are more sparse as $p \to \infty$."

Adding in level structure, we can define

$$\delta_N := \frac{\mathrm{mass}\left(\mathcal{Y}_0(N)^{ss}(K_N)\right)}{\mathrm{mass}\left(\mathcal{Y}_0(N)(K_N)\right)}$$

where $K_N \supseteq \mathbb{F}_p$ is the minimal field of definition of $\mathcal{X}_0(N)^{ss}$.

$\boxed{\text{Corollary (AP–K–P 2024+}\varepsilon\text{)}}$ For N prime to p,

$$(p-1) N \prod \left(1 + \frac{1}{\ }\right)$$

$$\delta_N \geq \frac{(p-1)N \prod_{\ell \mid N} \left(1+\frac{1}{\ell}\right)}{24 \, \# Y_0(N)\left(\mathbb{F}_{p^{12\phi(N)}}\right)} .$$

totient of N

Work in progress:

- Compute mass $\left(Y_0(N)\left(\mathbb{F}_{p^{12\phi(N)}}\right)\right)$ explicitly, at least for the genus 0 $Y_0(N)$. This refines the lower bound for δ_N.

- Compute asymptotic for δ_N as $p \to \infty$.

$\boxed{\text{Mass Formulas in Higher Dimensions}}$

The Eichler–Deuring formula

$$\sum_{E \in \mathcal{M}_{1,1}^{ss}(k)} \frac{1}{|\mathrm{Aut}(E)|} = \frac{p-1}{24}$$

has been generalized to the superspecial locus $\mathcal{A}_g^{ssp} \subseteq \mathcal{A}_g$ for $g \geq 2$:

$$\boxed{\text{Theorem (Hashimoto–Ibukiyama 1980, Ekedahl 1987)}}$$

For any p, $g \geq 1$,

$$\mathrm{mass}(\mathcal{A}_g^{ssp}(\overline{\mathbb{F}}_p)) = \frac{(-1)^{g(g+1)/2}}{2^g} \prod_{j=1}^{g} \zeta(1-2j) \prod_{k=1}^{g} (p^k + (-1)^k).$$

$\boxed{\text{Ex}}$ $\operatorname{mass}\left(A_2^{ssp}(\overline{\mathbb{F}}_p)\right) = \dfrac{(p-1)(p^2+1)}{5760}$

$\boxed{\text{Ex}}$ $\operatorname{mass}\left(A_3^{ssp}(\overline{\mathbb{F}}_p)\right) = \dfrac{(p-1)(p^2+1)(p^3-1)}{2903040}$

For other strata in the supersingular locus, mass formulas have been found by Yu (2003), Yu–Yu (2009), Karemaker–Tobuko–Yu (2021).

Joint work in progress with E. Assaf: Recover these mass formulas using stacky point counts of vanishing loci of appropriate mod p Siegel modular forms.

Theorem (Assaf – K. 2024 + ε) For $g = 2$,

there are mod p Siegel modular forms

$$\mathrm{Ha}_1 \in H^0(\mathcal{A}_2, \mathcal{L}^{\otimes p-1}) = M_{p-1}(\overline{\mathbb{F}}_p)$$

and $$\mathrm{Ha}_2 \in H^0(\mathcal{A}_2, \mathcal{L}^{\otimes p^2+1}) = M_{p^2+1}(\overline{\mathbb{F}}_p)$$

(partial Hasse invariants)

such that $\mathrm{mass}(\mathcal{A}_2^{ssp}(\overline{\mathbb{F}}_p))$ can be computed in terms of the stacky vanishing loci of Ha_1 and Ha_2.

Future directions:

- Use partial Hasse invariants $\mathrm{Ha}_1, \ldots, \mathrm{Ha}_g$ to compute the mass of $\mathcal{A}_g^{ssp}(\overline{\mathbb{F}}_p)$.

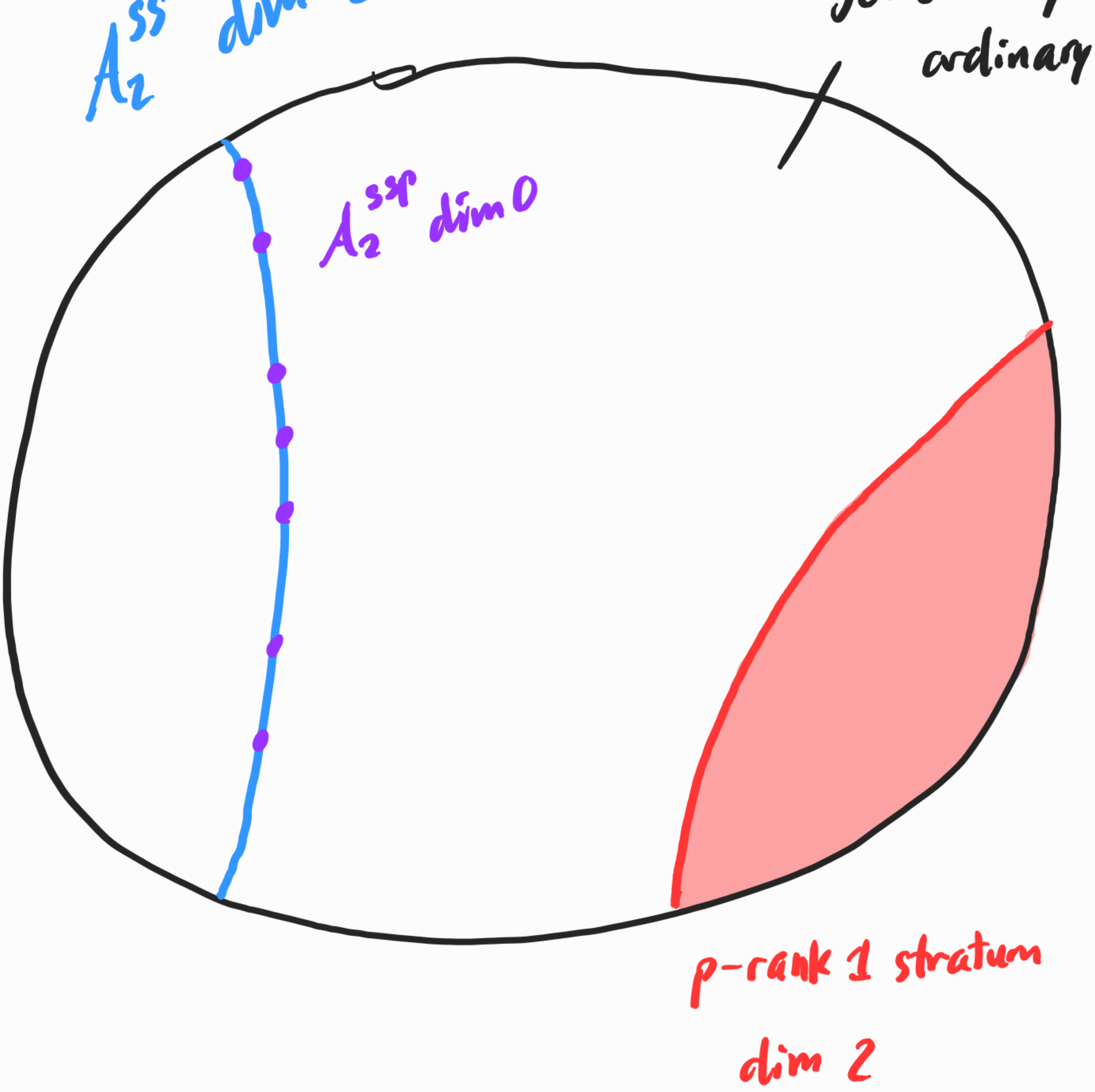
- Use Siegel modular forms to compute the masses of other strata in A_g^{ss}.

- Add level structure $\rightsquigarrow$ asymptotics ?

$$\boxed{\text{THANK YOU! } \quad \text{QUESTIONS ?}}$$

$1 \longleftrightarrow V(H_{a_1})$

generically



A_2 dim 3