ETAG 7/1/2020

Zeta functions in number theory, algebraic geometry and beyond!

Warmup: In small groups,

- recall some examples of zeta functions in any areas of math
- if possible, give precise definitions
- discuss any known connections between your examples

  e.g. is one a special case of another?

Examples

① $\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}$

② $K/\mathbb{Q}$ # field

$$\zeta_K(s) = \sum_{\substack{\text{ideals} \\ \mathfrak{a} \subset \mathcal{O}_K}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}$$

③ Function fields $\longleftrightarrow$ alg. curve

K tr.deg 1 $\qquad C / \mathbb{F}_q$

$\#C(\mathbb{F}_q)$ = number of $\mathbb{F}_q$-pts. of C

$$Z(C,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} t^n\right]$$

More generally, this can be defined (exactly the same) for variety / $\mathbb{F}_q$

(Hasse-Weil zeta function)

④ Motivic zeta function

$K_0(\mathrm{Var}_k)$ = Gr. gp. of k-varieties

$$:= \langle [X] \rangle \Big/ \left(\begin{matrix} [X]-[U]-[X\setminus U] \\ \text{if } U \subseteq_{\text{open}} X \end{matrix} \right)$$

$$\boxed{X = U \cup (X \setminus U)}$$

Kapranov's $Z_{mot}(X,t)$ is:

$$Z_{mot}(X,t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n X]\, t^n$$

$K_0(\mathrm{Var}_k)[[t]]$

$$\mathrm{Sym}^0 X = \mathrm{Spec}\, \mathbb{k} = *$$

$$\mathrm{Sym}^n X = X^n / \Sigma_n$$

$X = C$, $\# \mathrm{Sym}^n X(\mathbb{F}_q) = \# X(\mathbb{F}_{q^n})$

$$\# Z_{mot}(X,t) = \sum_{n=0}^{\infty} \underbrace{\# X(\mathbb{F}_{q^n})}\, t^n$$

$$\vdots$$

$$= Z_{HW}(X,t).$$

Zeta functions for posets

Classical setup: let $(P, \leq)$ be a poset and define an **interval** to be: $[x,y] = \{z \in P \mid x \leq z \leq y\}$.

Ex $(\mathbb{N}, \leq)$, $[x,y]$ = the usual interval

Ex $(\mathbb{N}, |)$, $[x,y] = \{d : x | d, d | y\}$

DEF For a poset $(P, \leq)$, define its **incidence coalgebra** $C(P)$ to be the free k-vector space on intervals $[x,y]$ with comultiplication

$$\Delta : C(P) \longrightarrow C(P) \otimes C(P)$$

$$[x,y] \longmapsto \sum_{z \in [x,y]} [x,z] \otimes [z,y]$$

and counit

$$\delta : C(P) \longrightarrow k$$

$$[x,y] \longmapsto \delta_{xy} = \begin{cases} 1, & x=y \\ 0, & x\neq y. \end{cases}$$

DEF The **incidence algebra** of $(P, \leq)$ is the vector space

$$I(P) = \mathrm{Hom}_k(C(P), k)$$

with multiplication

$$I(P) \otimes I(P) \longrightarrow I(P)$$
$$\varphi \otimes \psi \longmapsto (\varphi * \psi)([x,y]) := \sum_{z \in [x,y]} \varphi([x,z])\psi([z,y])$$

Assumption: $(P, \leq)$ is locally finite, i.e. all intervals $[x,y]$ are finite.

To any incidence algebra $I(P)$, there

In any incidence algebra $I(P)$ there

are distinguished elements:

the zeta function $\zeta : C(P) \to k$

$$[x,y] \mapsto 1$$

the Möbius function

$$\mu : C(P) \to k$$

$$[x,y] \mapsto \begin{cases} 1, & x = y \\ -\sum\limits_{z \in [x,y)} \mu([x,z]), & x \neq y \end{cases}$$

recursive

which satisfy

$$\zeta * \mu = \delta = \mu * \zeta$$

(i.e. ζ and μ are convolution

inverses in $I(P)$).

Ex. Let $(P, \leq) = (\mathbb{N}, |)$

Ex Let ...

For "arithmetic functions" $f: \mathbb{N} \to \mathbb{C}$ can construct a Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$.

e.g. the Riemann zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Dirichlet series attached to

$$\begin{aligned} \zeta : \mathbb{N} &\longrightarrow \mathbb{C} \\ n &\longmapsto 1 \end{aligned}$$

e.g. the Möbius series

$$\mu_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

Fact (Möbius inversion)

$$\mu_{\mathbb{Q}}(s) = \zeta_{\mathbb{Q}}(s)^{-1}$$

Idea: $F(s) = \sum \frac{f(n)}{n^s}$

$$G(s) = \sum \frac{g(n)}{n^s}$$

$$F(s)G(s) = \sum \frac{(f*g)(n)}{n^s}$$

where $(f*g)(n) = \underbrace{\sum_{ij=n} f(i)g(j)}_{\text{product in the incidence algebra } I(\mathbb{N},|)}$

Subtlety: this follows from the poset-level formula after taking the reduced incidence algebra

$[x,y] \longmapsto [1, \frac{y}{x}]$.

Fact: can deduce the Euler product

$$\zeta_{\mathbb{Q}}(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

from a decomposition of posets

$$(\mathbb{N}, |) \cong \prod_p (\{p^k\}, |) \cong \prod_p (\mathbb{N}_0, \leq)$$

$$n = \prod p_i^{k_i} \longleftrightarrow (p_i^{k_i}) \longleftrightarrow (k_i)$$

$\boxed{\text{"... and beyond": incidence algebras of decomposition spaces}}$

(due to Gálvez-Carrillo — Kock — Tonks)

Idea: incidence algebras don't just

come from posets (or more generally, categories), but from <u>higher homotopy theoretic structure</u>.

Recall: a simplicial set is a functor $X : \Delta^{op} \longrightarrow Set$, i.e.

cat. of comb. simplices

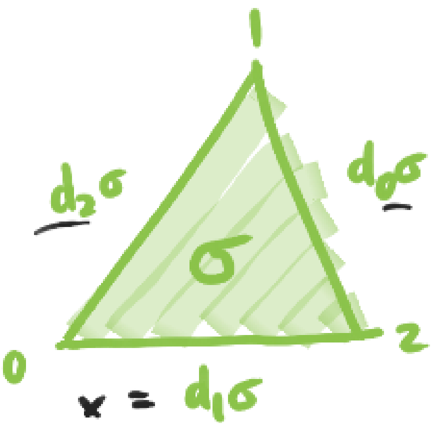
$$X_0 \leftrightarrows X_1 \leftrightarrows X_2 \leftrightarrows \cdots$$

$\boxed{\text{DEF}}$ If X is a (locally finite) simplicial set, its incidence coalgebra is the k-vector space is the k-vector space

$$C(X) = \bigoplus_{x \in X_1} k$$

with comultiplication

$$\Delta : C(X) \longrightarrow C(X) \otimes C(X)$$
$$[x] \longmapsto \sum_{\substack{\sigma \in X_2 \\ d_1\sigma = x}} \underline{d_2\sigma} \otimes \underline{d_0\sigma}$$



Fact / "definition" $C(X)$ is a coassociative, counital k-coalgebra exactly when X is a decomposition set (aka a decomp

2-Segal set. generalizing of $N(\mathcal{E})$

Epilogue: "It's never too late to topologize"

Gálvez-Carillo — Kock — Tonks (and independently Dyckerhoff-Kapranov) generalize decomposition sets (resp. 2-Segal sets) to decomposition spaces (resp. 2-Segal spaces).

$$\underline{\text{sets}} \rightsquigarrow \underline{\text{spaces}}$$

$$X : \Delta^{op} \to \mathrm{Set} \rightsquigarrow X : \Delta^{op} \to \mathcal{C}$$

vector space with basis $B \in \mathrm{Set}$ $\rightsquigarrow$ objects in the slice category $\mathcal{C}/B$

$$C(X) = \bigoplus_{x \in X_1} k \rightsquigarrow C(X) = \mathcal{C}/X_1$$

"free v.s. on X_1"

$$\Delta : C(X) \to C(X) \otimes C(X)$$

$$[x] \mapsto \sum_{d_1\sigma = x} d_2\sigma \otimes d_0\sigma \rightsquigarrow \Delta : \mathcal{C}/X_1 \to \mathcal{C}/X_1 \times X_1$$

induced by

$$\begin{tikzcd} & X_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & \\ X_1 & & X_1 \times X_1 \end{tikzcd}$$

$$I(X) = \mathrm{Hom}_k(C(X), k) \rightsquigarrow I(X) = \mathrm{Fun}(\mathcal{C}/X_1, \mathcal{C})$$

$$\varphi : [x] \mapsto 1 \rightsquigarrow \mathcal{C}/X_1 \xrightarrow{\varphi} \mathcal{C}$$

induced by

$$\begin{tikzcd} & X_1 \arrow[dl, "id"'] \arrow[dr] & \\ X_1 & & * \end{tikzcd}$$

To recover the classical theory, take $\mathcal{C} = \mathrm{Set}$ and apply cardinality $X \longleftrightarrow |X|$.

When $\mathcal{C}$ = Top, ∞-Cat., etc.,

can take homotopy cardinality.