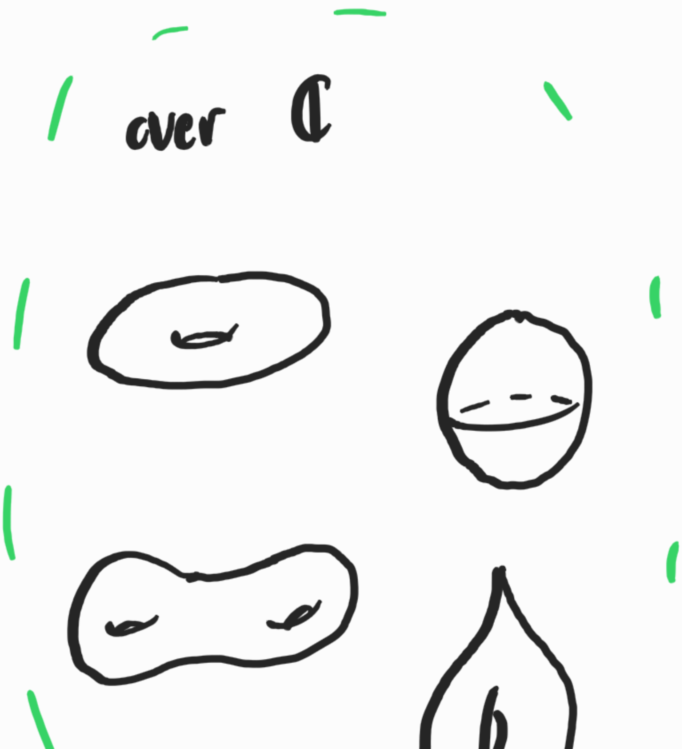
Moduli of Wildly Ramified Covers of Curves

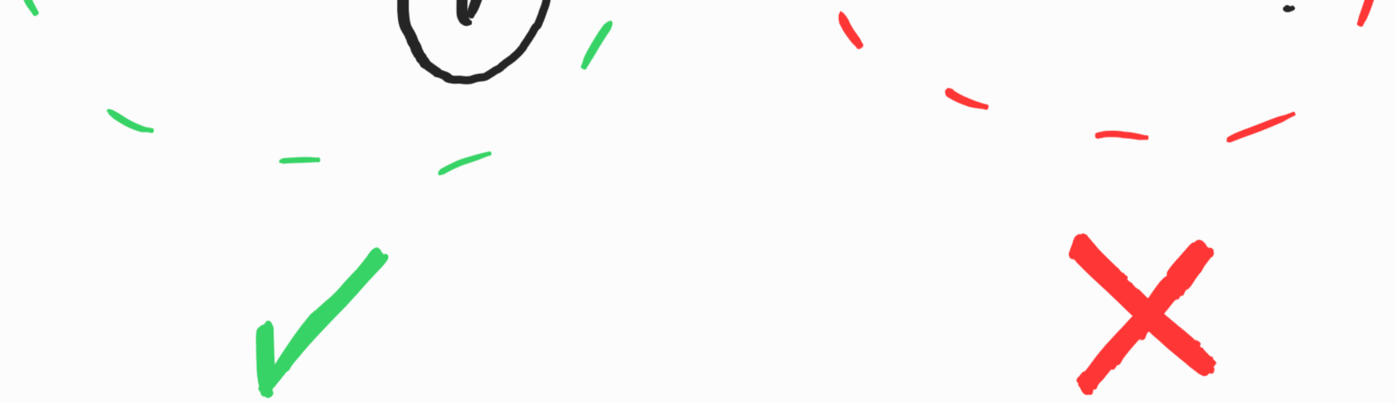
Andrew Kobin (UC Santa Cruz)

akobin@ucsc.edu / andrewkobin.com

some joint work in progress with V. Thatte

Introduction





The key in many situations is that structures (varieties, schemes, algebraic groups) have a rich topology over $\mathbb{C}$, which they lack over other fields/rings.

In arithmetic geometry, we want to leverage the geometry of these spaces to solve difficult counting problems over $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{Q}_p$, $\mathbb{Z}$, $\mathbb{Q}(\sqrt{D})$, $\mathbb{F}_q((t))$, etc.

Many ways to replace the usual tools from topology with algebraic tools.

$\boxed{\text{Ex}}$ The notion of topological cover has an algebraic analogue: an **étale cover** of varieties

$$V \longrightarrow U$$

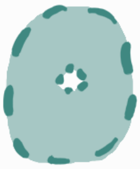
is one that behaves very much like a local homeomorphism between manifolds. (Rigorously, a map is étale if it is flat and unramified).

$\boxed{\text{Ex}}$ In parallel with covering theory in topology, there is a notion of étale fundamental group $\pi_1^{ét}(U)$ whose finite quotients correspond to finite étale covers of U.

Some examples over $\mathbb{C}$:

 $\pi_1^{ét}(\mathbb{P}^1_{\mathbb{C}}) = 1$ (Riemann sphere is simply connected)

 $\pi_1^{ét}(\mathbb{A}^1_{\mathbb{C}}) = 1$ (no unram. covers of affine line



$$\pi_1^{ét}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) = \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$$

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & & x \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \setminus \{0\} & & x^n \end{array}$$

genus g

In general, if $U = X \setminus \{x_1, \dots, x_n\}$ is a punctured Riemann surface, then

$$\pi_1^{ét}(U) \cong \widehat{\pi_1^{top}(\Sigma_{g,n})}$$

sfc. w/ genus g + n punct's

However, over a field k of characteristic

$p > 0$, things are not so nice...

$$\pi_1^{et}(\mathbb{P}^1_k) = 1$$ (the projective line is still "simply connected")

$$\pi_1^{et}(\mathbb{A}^1_k) = $$ ¯_(ツ)_/¯

By a theorem of Raynaud (1994), we know the finite quotients of $\pi_1^{et}(\mathbb{A}^1_k)$.

quasi-p, $\mathbb{Z}/m\mathbb{Z}$ $(p,m)=1$, some semidirect products

Key idea: étale covers of $\mathbb{A}^1$ come from ramified covers $Y \to \mathbb{P}^1$ that are branched at ∞.

When a point in the fibre of $Y \to \mathbb{P}^1$

over ∞ has stabilizer group whose order is divisible by p. We say the cover is **wildly ramified**.

(or just **wild**)

Artin-Schreier Covers of Curves

The basic building blocks of the wild part of $\pi_1^{et}(\mathbb{A}^1)$ are the $\mathbb{Z}/p\mathbb{Z}$ quotients, which come from the **Artin-Schreier covers** of $\mathbb{P}^1$.

In general, a (ramified) cover of curves

$$Y \longrightarrow \mathbb{P}^1$$

corresponds to a (ramified) field extension

$$k(Y)/k(t) =: L/K.$$

An Artin-Schreier cover is one that induces a Galois extension L/K with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/p\mathbb{Z}$.

Facts: (1) Every AS cover is given by an equation $y^p - y = f(t)$.

(2) The integer $j = \min\{\mathrm{ord}_\infty(f) \mid \text{all such } f\}$,

the ramification jump of $Y \to \mathbb{P}^1$,

is an invariant of the cover.

(3) In local coordinates, $Y \to \mathbb{P}^1$

is given by $y^p - y = t^{-j}$

with $p \nmid j$, as long as k is <u>perfect</u>.

(4) The genus of Y (as an alg. curve)

is $g(Y) = \frac{(p-1)(j-1)}{2}$.

(5) In my thesis, I relate AS covers

of curves (including $Y \to X$, $X \neq \mathbb{P}^1$)

to the theory of stacky curves

by way of Artin-Schreier root stacks.

Ex

$$Y : \quad y^p - y = \frac{1}{t^p}$$

$$\downarrow$$

$$\mathbb{P}^1$$

$$j = \min\{\mathrm{ord}_\infty(f)\} < p$$

$$(y')^p - y' = f$$

$$\underbrace{\left(y + \frac{1}{t}\right)^p} - \left(y + \frac{1}{t}\right) = \frac{1}{t^p}$$

$$y^p + \frac{1}{t^p} - y - \frac{1}{t} = \frac{1}{t^p}$$

$$y^p - y = \frac{1}{t}$$

$$j = 1$$

In general, if k is not perfect, work of K. Kato and V. Thatte provide a preferred $f \in K$ for the local equation

equation

$$Y : y^p - y = f(t)$$

(called "best f") whose pole order at ∞ coincides with the Swan conductor of $Y/\mathbb{P}^1$ at ∞.

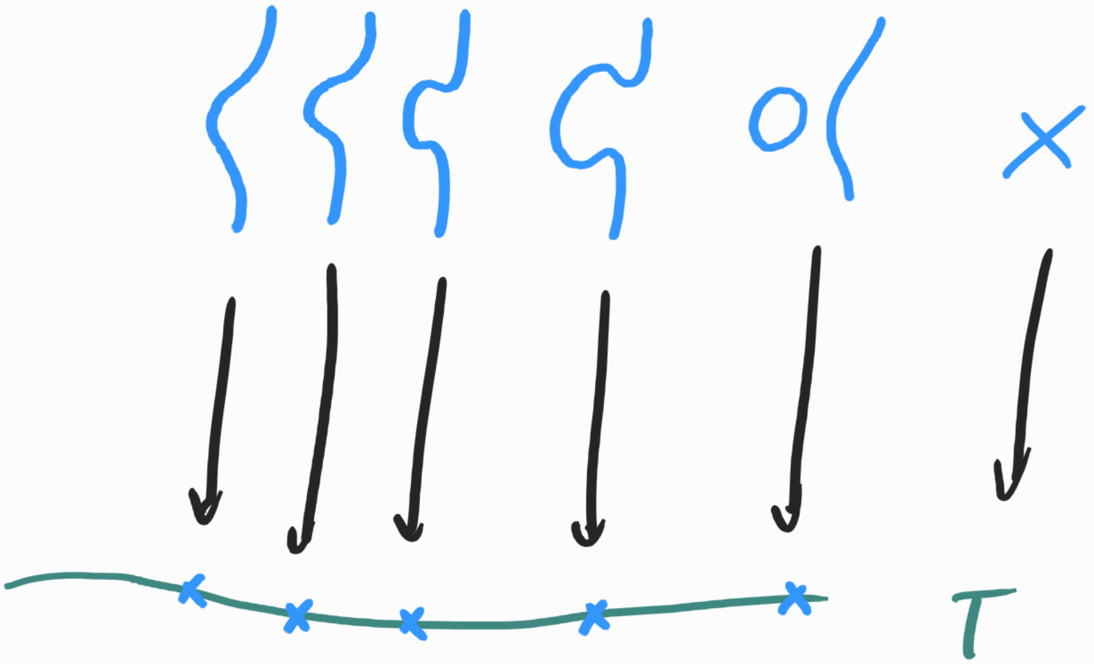
$\boxed{\text{Moduli of Curves}}$

Suppose we want to study these covers $Y \longrightarrow \mathbb{P}^1$ in families:

- a family of curves over a scheme T (or a T-curve) is a flat morphism

$$X \longrightarrow T$$

such that each fibre X_t is a (smooth projective) curve.



- a family of covers of curves (or a cover of T-curves) is a commutative triangle

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X \arrow[d] \\ & T \end{tikzcd}$$

where all the fibres $Y_t \to X_t$ are covers of curves.

Let $AScov_g^j \subseteq \mathcal{M}_g$ denote the moduli stack of Artin-Schreier covers of genus g and ramification jump j.

That is, for a scheme T,

$\mathrm{AScov}_g^j(T)$ is the groupoid

of Artin-Schreier covers

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[dr] & \mathbb{P}^1 \arrow[d] \\ & T \end{tikzcd}$$ with

- $g(Y_t) = g$
- φ_t is ramified at ∞ with ramification jump j.

When $T = \mathrm{Spec}\, k$ for k an algebraically closed field, work of Pries - Zhu provides an explicit parametrization of $\mathrm{AScov}_g^j(k)$

using combinatorics.

Current work in progress w/ Thatte:

Combine "best f", the Swan

conductor and my work on

Artin-Schreier root stacks to

provide a more general description

of AScov_g^j ...

... and beyond!

Thanks for your attention!

Questions?

$$y^p - y = t^{-j} = f$$

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, "\mathbb{Z}/p"'] & & \mathbb{P}^1 \arrow[dd, "\gamma"] \\
& \mathbb{P}(1,j) \arrow[ur] & \\
\mathbb{P}^1 \arrow[rr, "f"] \arrow[dr, bend right] & & \mathbb{P}^1 \arrow[dl, shift right] \\
& \mathbb{P}(1,j) \arrow[ur, shift right] &
\end{tikzcd}$$

unique
num. j

$$\mathcal{A}\mathrm{Scov}_g^j(k) = \left\{ \begin{tikzcd} Y \arrow[r] \arrow[d] & \mathbb{P}(1,j) \arrow[d] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}(1,j) \end{tikzcd} \right\}$$

$$\{\ \mathbb{P} \longrightarrow \mathbb{P}(1,j)\ \}$$

$$\cong$$

$$\mathrm{Hilb}^{ss}(k)$$