Due Date: Thursday, March 2 at 11:59PM EST

Carefully read and provide solutions to the problems below, showing all work required to justify any conclusions you make. You are encouraged to collaborate with your classmates, but all solutions turned in should be your own work. If you do collaborate, please record the names of those other students on your submitted work. Finally, your work should be submitted as a PDF on Gradescope before the listed due date.

Textbook problems: 5.1, 5.3

Problem 1. In this problem, you will construct a finite field with $5^5 = 3125$ elements.

(a) Show that $F = \mathbb{F}_5[x]/(x^5 - x + 1)$ is a field. *Hint: HW 4, Problem 2(b) might help.*

(b) Show that $x^5 - x + 1$ has a root in F, call it α.

(c) Prove that F is a 5-dimensional vector space over $\mathbb{F}_5$ by showing that

$$F = \{a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 \mid a, b, c, d, e \in \mathbb{F}_5\}$$

and showing that $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ are linearly independent over $\mathbb{F}_5$.

Problem 2. Let $\zeta = e^{2\pi i/n}$ for a fixed integer $n \geq 2$.

(a) Prove that $1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1} = 0$.

(b) When $n = 5$, use (a) to show that the complex number $\zeta + \bar{\zeta}$ is a solution to the quadratic equation $z^2 + z - 1 = 0$.

(c) Use (b) to show that the real number $\cos\left(\frac{2\pi}{5}\right)$ is a solution to a quadratic equation $ax^2 + bx + c = 0$ for some $a, b, c \in \mathbb{R}$. Then solve this equation using the quadratic formula to compute $\cos\left(\frac{2\pi}{5}\right)$ exactly.

Problem 3. (Lecture 6.2, Exercise 1) Let $\mathbb{Q}(\zeta_n)$ be the nth cyclotomic extension of $\mathbb{Q}$.

(a) For $p \geq 3$ prime, prove that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$.

(b) Do you have a guess what $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ is for n composite?