Due Date: Thursday, October 27 at 5PM EDT

Carefully read and provide solutions to the problems below, showing all work required to justify any conclusions you make. You are encouraged to collaborate with your classmates, but all solutions turned in should be your own work. If you do collaborate, please record the names of those other students on your submitted work. Finally, your work should be submitted as a PDF on Gradescope before the listed due date.

Textbook problems: 22.1, 22.2, 22.3, 23.5

Problem 1. Suppose that $x \in \mathbb{N}$ is not divisible by 5 and $n = x^2 - 5$. Prove that every odd prime divisor of n is congruent to 1 or 4 mod 5.

Problem 2. Prove there are infinitely many primes $p \equiv 9 \pmod{10}$. *Hint: if $p_1, \ldots, p_r$ are all the primes congruent to 9 mod 10, define $x = 2p_1 \cdots p_r$ and show that $x^2 - 5$ must have a prime divisor $p \equiv 9 \pmod{10}$. Find a contradiction.*

Problem 3. Show that $\left(\frac{3}{p}\right) = 1$ for any prime $p \equiv 1 \pmod{12}$.

This won't be graded, but once you finish Problem 3, try to describe $\left(\frac{3}{p}\right)$ for all odd primes p and prove your formula.

Problem 4. (Lecture 9.2, Exercise 1) Check that the formula

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

is equivalent to the original statement of Quadratic Reciprocity from Lecture 9.1.

Problem 5. (Lecture 9.2, Exercise 2) Verify that for an odd prime p,

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

You may use the theorem we proved for $\left(\frac{2}{p}\right)$ in your proof; this problem is just asking you to verify this alternative formulation of that reciprocity law.