Due Date: Thursday, December 1 at 5PM EDT

Carefully read and provide solutions to the problems below, showing all work required to justify any conclusions you make. You are encouraged to collaborate with your classmates, but all solutions turned in should be your own work. If you do collaborate, please record the names of those other students on your submitted work. Finally, your work should be submitted as a PDF on Gradescope before the listed due date.

Textbook problems: 27.2, 29.4

Problem 1. Find all incongruent solutions to $x^5 \equiv 14 \pmod{101}$.

Problem 2. (a) (Exercise 1 from Lecture 13.2) Prove the product formula for the zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Hint: Use the Fundamental Theorem of Arithmetic.

(b) More generally, let $f(n)$ be a completely multiplicative arithmetic function. Show that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

has a product formula

$$F(s) = \prod_{p \text{ prime}} \frac{1}{1 - f(p)p^{-s}}.$$

Problem 3. Let $\phi(n)$ be the totient function and define its Dirichlet series

$$T(s) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$

Show that $T(s)$ satisfies the following relation with the zeta function $\zeta(s)$:

$$T(s) = \frac{\zeta(s-1)}{\zeta(s)}.$$

For this problem, don't worry about convergence.