Categorifying zeta and L-functions

Andrew J. Kobin

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Rethinking Number Theory

January 5, 2023

EMORY UNIVERSITY

Joint work with Jon Aycock

Introduction

This talk is based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments:	23 pages; minor changes and additional references added
Subjects:	**Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes:	11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as:	arXiv:2011.13903 **[math.NT]**
	(or arXiv:2011.13903v2 **[math.NT]** for this version)

and

Categorifying quadratic zeta functions

Jon Aycock, Andrew Kobin

The Dedekind zeta function of a quadratic number field factors as a product of the Riemann zeta function and the L-function of a quadratic Dirichlet character. We categorify this formula using objective linear algebra in the abstract incidence algebra of the division poset.

Comments:	27 pages
Subjects:	**Number Theory (math.NT)**
MSC classes:	11M06, 11M41, 18N50, 06A11, 16T10
Cite as:	arXiv:2205.06298 **[math.NT]**
	(or arXiv:2205.06298v1 **[math.NT]** for this version)

and generalizations to L-functions (work in progress with J. Aycock)

Team Zeta

Here's Jon!

and the rest of Team Zeta:

Karen Acquista

Changho Han

Alicia Lamarche

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

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Two Zeta Formulas

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where χ is the quadratic character cutting out $K/\mathbb{Q}$.

Analogously, for an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t)L(E,t).$$

Categorification Strategy

We can formalize algebraic properties of zeta functions in an incidence algebra of “arithmetic functions” using objective linear algebra.

Basic idea:

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mathbf{1}}{n^s}$$ Dirichlet series

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$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots) \in I_{\#}(\mathbb{Q})$ arithmetic functions

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Basic idea:

$$\mathbb{N} \xleftarrow{\ \mathrm{id}\ } \mathbb{N} \longrightarrow *$$

linear functors

$\big\downarrow$

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arithmetic functions

$\big\updownarrow$

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{\mathbf{1}}{n^s}$$

Dirichlet series

Categorification Strategy: Number Fields

We can formalize algebraic properties of zeta functions in an incidence algebra of "arithmetic functions" using objective linear algebra.

Basic idea for $K/\mathbb{Q}$:

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\mathbf{1}}{N(\mathfrak{a})^s} \qquad \text{Dirichlet series}$$

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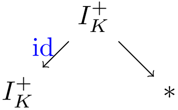

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Categorification Strategy: Curves over Finite Fields

We can formalize algebraic properties of zeta functions in an incidence algebra of “arithmetic functions” using objective linear algebra.

Basic idea for $C/\mathbb{F}_q$:

$$Z(C,t) = \sum_{\alpha \in Z_0^{\text{eff}}(C)} \mathbf{1} t^{\det(\alpha)}$$

power series

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↕

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power series

Is everything zeta? ¯_(ツ)_/¯

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

Goal: lift different zeta and L-functions to their “native” objective incidence algebras and connect them functorially using simplicial maps

e.g. for $K/\mathbb{Q}$ quadratic (Aycock–K., 2022):

$$\begin{array}{ccc} S(K) & & S(\chi) \\ & \searrow \quad \swarrow & \\ & S(\mathbb{Q}) & \end{array}$$

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$$

Decomposition Sets

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftrightarrows S_1 \leftrightarrows S_2 \cdots .$$

Example

Any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

Decomposition Sets

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \cdots S_2 \cdots .$$

Example

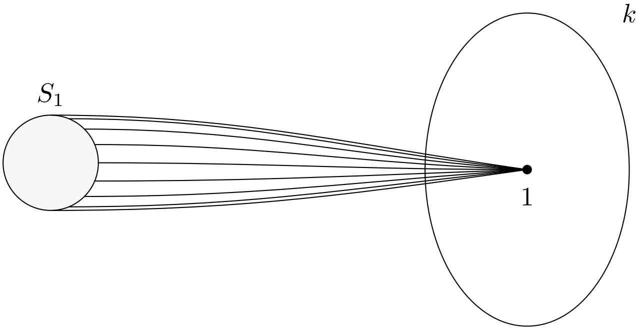
I_K^+ determines a simplicial set with:

- 0-simplices $=$ ideals $\mathfrak{a}$ in I_K^+
- 1-simplices $=$ divisibility $\mathfrak{b} \to \mathfrak{a} \iff \mathfrak{a} \mid \mathfrak{b}$
- 2-simplices $=$ decompositions $\mathfrak{b} \to \mathfrak{c} \to \mathfrak{a}$
- etc.

Decomposition Sets

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

For any decomposition set S, there is a distinguished **zeta function** $\zeta : x \mapsto 1$.



To bring L-functions into the mix, need **objective linear algebra**.

Objective Linear Algebra

Final ingredient: **objective linear algebra** ("linear algebra with sets"):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

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To recover vector spaces, take $V = k^B$ and take cardinalities.

Objective Linear Algebra

Let $S = I_K^+$ or $Z_0^{\text{eff}}(C)$. The **objective incidence algebra** of S is defined by:

Numerical	Objective
basis S_1	set S_1
vector space k^{S_1}	slice category $\text{Set}_{/S_1}$
$I_\#(S) = \text{Hom}(k^{S_1}, k)$	$I(S) := \text{Lin}(\text{Set}_{/S_1}, \text{Set})$

Here, an element $f \in I(S)$ is a linear functor $f = t_! s^* : \text{Set}_{/S_1} \to \text{Set}$ represented by a span

$$f = \left(\begin{array}{ccccc} & & M & & \\ & \swarrow^{s} & & \searrow^{t} & \\ S_1 & & & & * \end{array} \right)$$

Abstract Incidence Algebras

Here, an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(S_1 \xleftarrow{s} M \xrightarrow{t} * \right)$$

Example

The **zeta functor** is the element $\zeta \in I(K)$ represented by

$$\zeta = \left(S_1 \xleftarrow{\mathrm{id}} S_1 \longrightarrow * \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \begin{pmatrix} & & M & & \\ & \swarrow^{s} & & \searrow & \\ S_1 & & & & * \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} & & N & & \\ & \swarrow^{t} & & \searrow & \\ S_1 & & & & * \end{pmatrix}$$

the convolution $f * g \in I(S)$ is represented by

$$(f * g) = \begin{pmatrix} & & & P & & & \\ & & \swarrow & & \searrow & & \\ & S_2 & & & & M \times N & \\ \swarrow^{d_1} & & \searrow^{(d_2, d_0)} & & \swarrow_{s \times t} & & \searrow \\ S_1 & & & S_1 \times S_1 & & & * \end{pmatrix}$$

Quadratic Number Fields

Recall the formula

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$$

Theorem (Aycock–K., '22)

In the reduced incidence algebra $\widetilde{I}(\mathbb{Q}) := \widetilde{I}(\mathbb{N}, |)$, there is an equivalence of linear functors

$$N_* \zeta_K + \zeta_{\mathbb{Q}} * L(\chi)^- \cong \zeta_{\mathbb{Q}} * L(\chi)^+$$

where $N : I_K^+ \to \mathbb{N}$ is the norm map and $L(\chi)^+$ and $L(\chi)^-$ are functors in $\widetilde{I}(\mathbb{N})$.

Taking cardinalities, the formula reads

$$N_* \zeta_K = \zeta_{\mathbb{Q}} * (L(\chi)^+ - L(\chi)^-) = \zeta_{\mathbb{Q}} * L(\chi).$$

Elliptic Curves

And the geometric version:

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

Theorem (Aycock–K., '23+ϵ)

In the reduced incidence algebra $\widetilde{I}(E) := \widetilde{I}(Z_0^{\mathrm{eff}}(E))$, there is an equivalence of linear functors

$$\pi_* \zeta_E + \zeta_{\mathbb{P}^1} * L(E)^- \cong \zeta_{\mathbb{P}^1} * L(E)^+$$

where $\pi : E \to \mathbb{P}^1$ is a fixed double cover and $L(E)^+$ and $L(E)^-$ are functors in $\widetilde{I}(\mathbb{P}^1)$.

Pushing forward to $\widetilde{I}(\operatorname{Spec} \mathbb{F}_q)$ and taking cardinalities, it reads

$$\pi_* \zeta_E = \pi_* \zeta_{\mathbb{P}^1} * (L(E)^+ - L(E)^-) = \pi_* \zeta_{\mathbb{P}^1} * L(E).$$

Highlights and Dreams for the Future

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- It's pretty fun to prove things!

Future work / work in progress:

- Motivic zeta functions $Z_{mot}(X, t)$
- Construct $\zeta_{\mathcal{X}}$ for an algebraic stack $\mathcal{X}$
- Lift L-functions $L(V)$ of Galois representations
- Archimedean zeta functions

Thank you!