Root Stacks in Characteristic p

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Explicit Methods in Characteristic p at JMM 2020

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Introduction

Common problem: all sorts of information is lost when we consider quotient objects and/or singular objects.

However, many of these properties are recovered when we pass to the language of *orbifolds*.

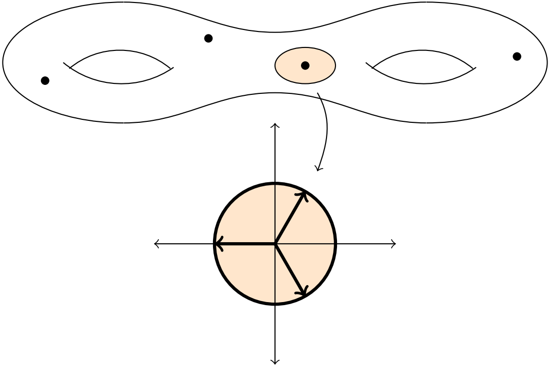
e.g. invariants of modular curves $X_0(N)$ have geometric meaning when $X_0(N)$ is treated as an orbifold.

Goal: Classify stacky curves (= orbifold curves) in char. p (thesis work - preprint available!)

Complex Orbifolds

Definition

A **complex orbifold** is a topological space admitting an atlas $\{U_i\}$ where each $U_i \cong \mathbb{C}^n/G_i$ for a finite group G_i, satisfying compatibility conditions (think: manifold atlas but with extra info).

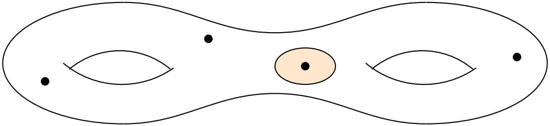


Algebraic Orbifolds

There's also a version of orbifold in the 'algebraic category'

Technical definition: blah blah blah Deligne–Mumford stack

Non-technical definition: smooth variety/scheme with a finite stabilizer group attached at each point



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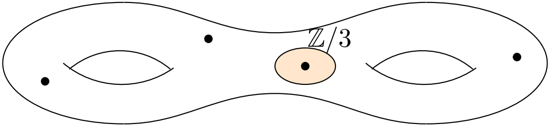

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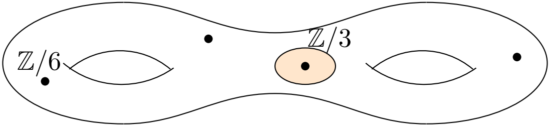

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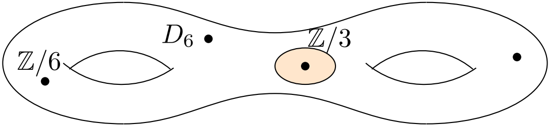

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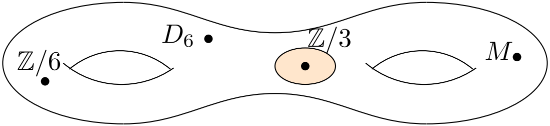

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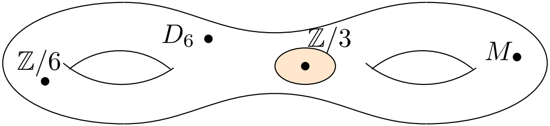
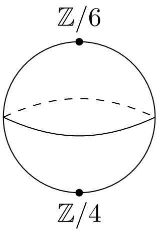


Focus on orbifold curves for the rest of the talk

An Example

Example

The (compactifed) moduli space of complex elliptic curves is an orbifold $\mathbb{P}^1$ with a generic $\mathbb{Z}/2$ and a special $\mathbb{Z}/4$ and $\mathbb{Z}/6$.



Root Stacks

Key fact: over $\mathbb{C}$, all stabilizers are *cyclic*.

So orbifolds can be locally modeled by a *root stack*: charts look like

$$U \cong [\operatorname{Spec} A/\mu_n]$$

where $A = k[y]/(y^n - x_0)$ and μ_n is the group of nth roots of unity.

(Think: degree n branched cover mod μ_n-action, but remember the action.)

Root Stacks

More rigorously:

Definition (Cadman, Abramovich–Olsson–Vistoli)

Let X be a variety and $L \to X$ a line bundle with section $s : X \to L$. The **nth root stack** of X along (L, s) is the fibre product

$$\begin{array}{ccccc} \sqrt[n]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & & x \\ \downarrow & & \downarrow & & \big\downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] & & x^n \end{array}$$

Here, $[\mathbb{A}^1/\mathbb{G}_m]$ is the classifying stack for pairs (L, s).

Interpretation: $\sqrt[n]{(L,s)/X}$ admits a canonical tensor nth root of (L, s), i.e. (M, t) such that $M^{\otimes n} = L$ and $t^n = s$ (after pullback).

Root Stacks

Theorem (Geraschenko–Satriano '15)

Every smooth separated **tame** *Deligne–Mumford stack of finite type with trivial generic stabilizer is** *a root stack over its coarse space.*

Corollary

Tame *stacky curves are completely described by their coarse space and a finite list of numbers corresponding to the orders of cyclic stabilizers at a finite number of stacky points.*

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What happens with **wild** stacky curves in char. p?

Interlude: Why Characteristic p?

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$$|\mathbb{F}_p| << |\mathbb{C}|$$

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$$(x+y)^p = x^p + y^p \text{ (obviously)}$$

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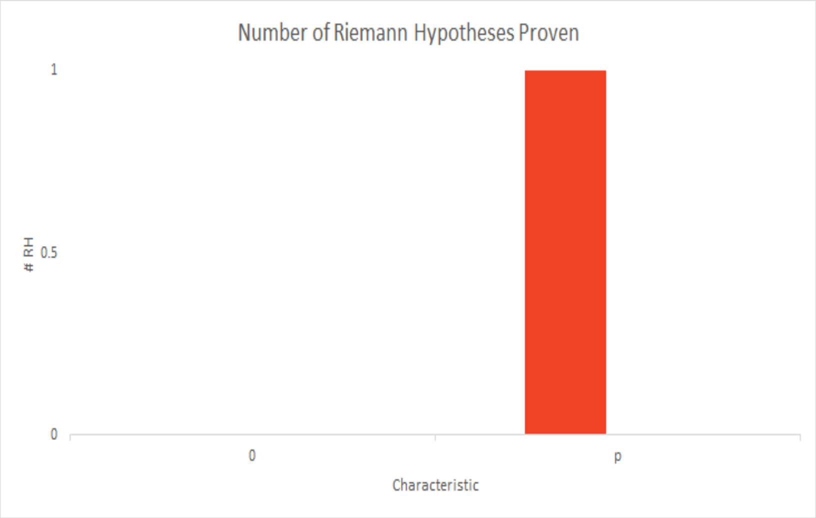

Interlude: Why Characteristic p?

Things get **wild** (e.g. $\pi_1(\mathbb{A}^1) =$ ¯_(ツ)_/¯)

Artin–Schreier Root Stacks

In trying to classify **wild** stacky curves in char. p, we face the following problems:

(1) Stabilizer groups need not be cyclic (or even abelian)

(2) Cyclic $\mathbb{Z}/p^n\mathbb{Z}$-covers of curves occur in families

(3) Root stacks don't work

- Finding $M^{\otimes p}$ is a problem
- $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m], x \mapsto x^p$ is a problem

Key case: cyclic $\mathbb{Z}/p\mathbb{Z}$-stabilizers

Artin–Schreier Root Stacks

Idea: replace **tame** cyclic covers $y^n = f(x)$ with **wild** cyclic covers $y^p - y = f(x)$.

More specifically: **Artin–Schreier theory** classifies cyclic degree p-covers of curves in terms of the **ramification jump** (e.g. if $f(x) = x^m$ then m is the jump).

This suggests introducing wild stacky structure using the local model

$$U = [\operatorname{Spec} A/(\mathbb{Z}/p)]$$

where $A = k[y]/(y^p - y - f(x))$ and $\mathbb{Z}/p$ acts additively.

Artin–Schreier Root Stacks

How do we do it?

$$\begin{array}{ccc} \sqrt[n]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array} \qquad\qquad \begin{array}{c} x \\ \downarrow \\ x^n \end{array}$$

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Artin–Schreier Root Stacks

How do we do it?

$$\begin{array}{ccc} \wp_1^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}^1/\mathbb{G}_a] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}^1/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \mapsto \\ [u^p, v^p - vu^{p-1}] \end{array}$$

Artin–Schreier Root Stacks

Definition (K.)

Fix $m \geq 1$. Let X be a variety, $L \to X$ a line bundle and $s : X \to L$ and $f : X \to L^{\otimes m}$ two sections not vanishing simultaneously. The **Artin–Schreier root stack** of X with jump m along (L, s, f) is the fibre product

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(1,m)/\mathbb{G}_a] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}(1,m)/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \downarrow \\ [u^p, v^p - vu^{m(p-1)}] \end{array}$$

where

- $\mathbb{P}(1, m)$ is the weighted projective line with weights $(1, m)$
- $\mathbb{G}_a = (k, +)$, acting additively
- $[\mathbb{P}(1, m)/\mathbb{G}_a]$ is the classifying stack for triples (L, s, f) up to the principal part of f.

Artin–Schreier Root Stacks

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Interpretation: $\wp_m^{-1}((L,s,f)/X)$ admits a canonical pth root of L, i.e. a line bundle M such that $M^{\otimes p} = L$, and an AS root of s.

Artin–Schreier Root Stacks

Key example:

Example (K.)

Consider the AS cover

$$\begin{array}{c} Y \;:\; y^p - y = x^{-m} \\ \mathbb{Z}/p \big\downarrow \\ \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1] \end{array}$$

Then $\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p)]$.

In general, every AS root stack is étale-locally isomorphic to such an "elementary AS root stack".

Classification of (Some) Wild Stacky Curves

So let's classify us some wild stacky curves!

Theorem 1 (K.)

Every Galois cover of curves $\varphi : Y \to X$ with an inertia group $\mathbb{Z}/p$ factors étale-locally through an Artin–Schreier root stack $\wp_m^{-1}((L, s, f)/X)$.

Informal consequence: there are infinitely many non-isomorphic stacky curves over $\mathbb{P}^1$ with a single stacky point of order p.

This phenomenon only occurs in char. p.

Classification of (Some) Wild Stacky Curves

Main result:

Theorem 2 (K.)

Every stacky curve $\mathcal{X}$ with a stacky point of order p is étale-locally isomorphic to an Artin–Schreier root stack $\wp_m^{-1}((L,s,f)/U)$ over an open subscheme U of the coarse space of $\mathcal{X}$.

This even holds globally if $\mathcal{X}$ has coarse space $\mathbb{P}^1$:

Theorem 3 (K.)

If $\mathcal{X}$ has coarse space $\mathbb{P}^1$ and all stacky points of $\mathcal{X}$ have order p, then $\mathcal{X} \cong \wp_m^{-1}((L,s,f)/\mathbb{P}^1)$ for some $(m,p)=1$ and (L,s,f).

However, this fails in general.

Generalizations

What about $\mathbb{Z}/p^2$-covers, stacky points of order p^2, and beyond?

For cyclic stabilizer groups $\mathbb{Z}/p^n$, Artin–Schreier theory is subsumed by **Artin–Schreier–Witt theory**:

- AS covers $y^p - y = f(x)$ are replaced by Witt vector equations $\underline{\mathsf{y}}^p - \underline{\mathsf{y}} = (f_0(\underline{\mathsf{x}}), \ldots, f_n(\underline{\mathsf{x}}))$
- Covers are characterized by *sequences of ramification jumps*
- Local structure is $U = [\mathrm{Spec}\, A/(\mathbb{Z}/p^n)]$ where $A = k[\underline{\mathsf{y}}]/(\underline{\mathsf{y}}^p - \underline{\mathsf{y}} - \underline{\mathsf{f}})$

Generalizations

This local structure can be formally introduced using **Artin–Schreier–Witt root stacks**:

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(1,m)/\mathbb{G}_a] \\ \downarrow & & \downarrow \wp_m \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}(1,m)/\mathbb{G}_a] \end{array}$$

where

- $\mathbb{W}_n$ is the ring of length n Witt vectors
- $\overline{\mathbb{W}}_n(\bar{m})$ is a stacky compactification of $\mathbb{W}_n$ with weights $\bar{m} = (m_1, \ldots, m_n)$
- $[\overline{\mathbb{W}}_n(\bar{m})/\overline{\mathbb{W}}_n]$ classifies tuples $(L, s, f_0, \ldots, f_n)$ up to the principal part of the f_i.

(In progress) Final steps are to classify stacky curves with $\mathbb{Z}/p^n$-structure using this construction.

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Thank you!