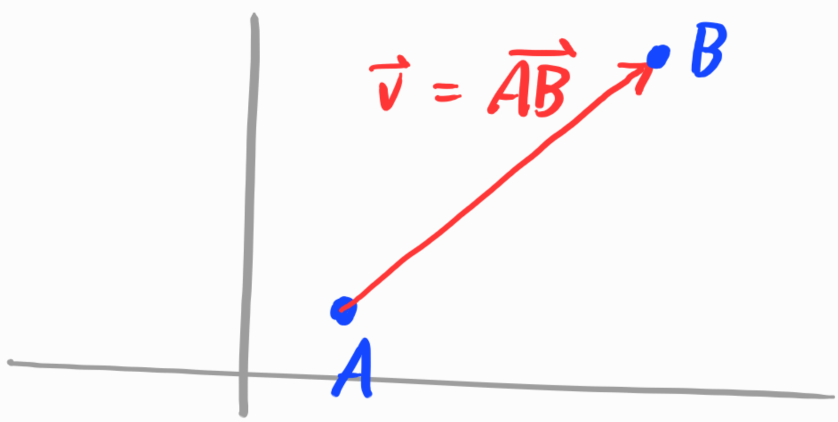
Lecture 12.2

To move around n-dimensional space and better study multivariable functions, we will use the notion of a vector.

$\boxed{\text{Def}}$ A vector in $\mathbb{R}^n$ is a choice of direction and magnitude, which is represented by a physical arrow $\vec{v}$ in $\mathbb{R}^n$:



A = the tail of $\vec{v}$
B = the head of $\vec{v}$
these specify the direction

$|\vec{v}| = |AB|$ specifies the magnitude,
(distance) or length

A vector in $\mathbb{R}^n$ is uniquely specified by a list of coordinates,

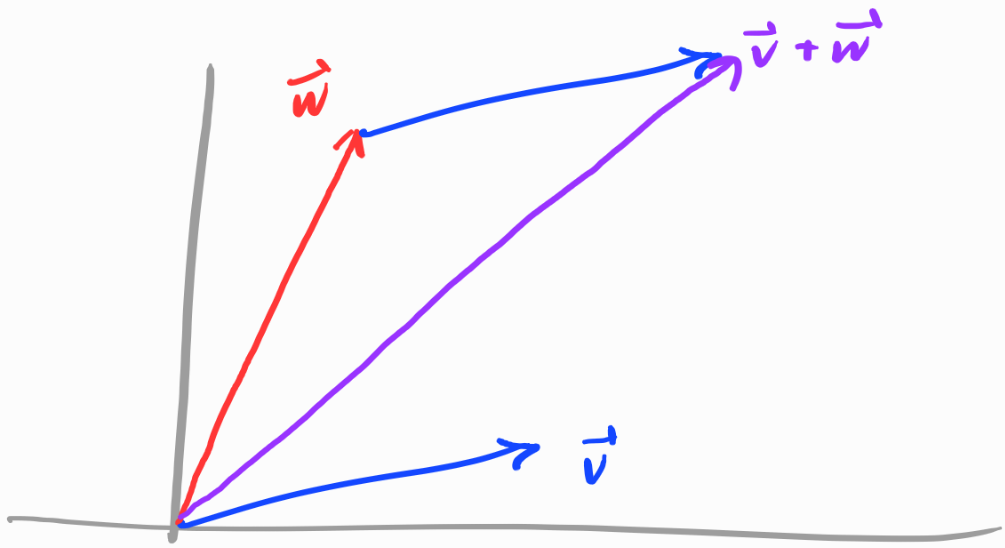
$$\vec{v} = \langle v_1, \ldots, v_n \rangle.$$

If $\vec{v} = \overrightarrow{AB}$ with $A = (a_1, \ldots, a_n)$
and $B = (b_1, \ldots, b_n)$

then $\vec{v} = \langle b_1 - a_1, \ldots, b_n - a_n \rangle.$

We can change and combine vectors to form new vectors:

- Two vectors can be added head-to-tail:



In coordinates,

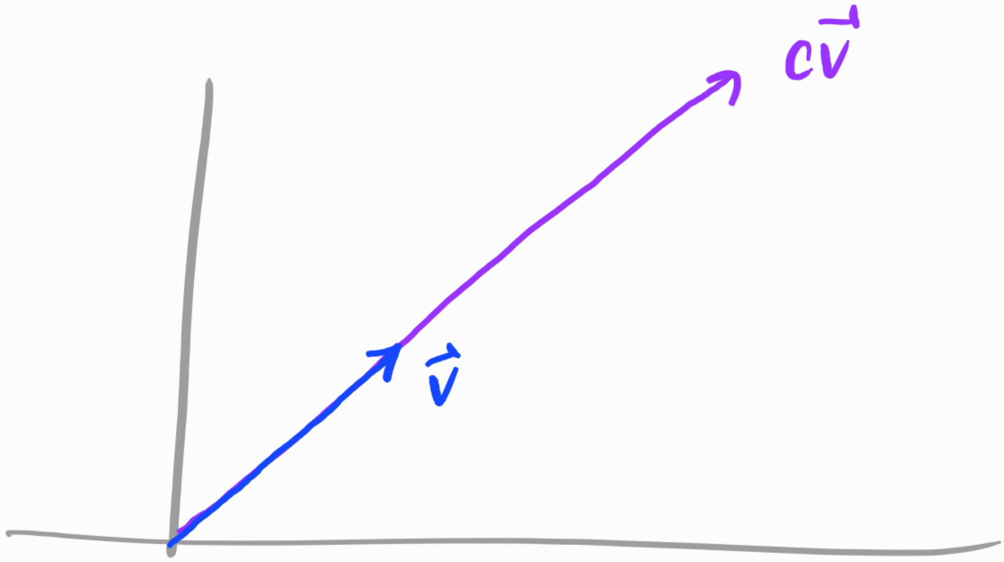
if $\vec{v} = \langle v_1, \ldots, v_n \rangle$

and $\vec{w} = \langle w_1, \ldots, w_n \rangle$

then $\vec{v} + \vec{w} = \langle v_1 + w_1, \ldots, v_n + w_n \rangle$

then $\vec{v} + \vec{w} = \langle v_1 + w_1, \dots, v_n + w_n \rangle$.

- A vector $\vec{v}$ can be scaled by a real number c:



In coordinates:

if $\vec{v} = \langle v_1, \dots, v_n \rangle$

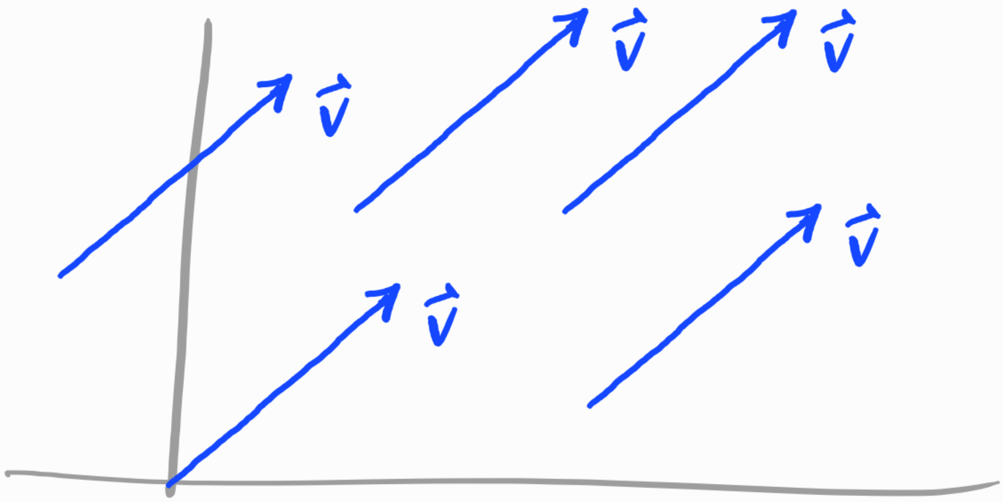
then $c\vec{v} = \langle cv_1, \dots, cv_n \rangle$.

$\boxed{\text{Ex}}$ Vectors are useful in the real

Ex Vectors are useful in the real world too:

- "head 3 miles south"
- the velocity of an airplane is a 3-dimensional vector
- force in physics is a vector, with a direction and strength (magnitude).

Note: a vector is independent of starting point, e.g.



all have the same direction and magnitude, so they're the same vector!

However, every vector can be visualized as starting at the origin, in which case its head has the same coord's as the vector itself.

$\boxed{\text{Ex}}$ Take $\vec{v} = \langle 4, 0, 3 \rangle$, $\vec{w} = \langle -2, 1, 5 \rangle$ in $\mathbb{R}^3$. Then

$$|\vec{v}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$|\vec{w}| = \sqrt{(-2)^2 + 1^2 + 5^2} = \sqrt{30}$$

$$\vec{v} + \vec{w} = \langle 4-2, 0+1, 3+5 \rangle = \langle 2, 1, 8 \rangle$$

$$\vec{v} - \vec{w} = \langle 4+2, 0-1, 3-5 \rangle = \langle 6, -1, -2 \rangle$$

$$2\vec{v} + 5\vec{w} = \langle 2\cdot 4, 2\cdot 0, 2\cdot 3 \rangle + \langle 5\cdot -2, 5\cdot 1, 5\cdot 5 \rangle$$

$$= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle$$

$$= \langle -2, 5, 31 \rangle.$$

Here are some properties of vectors

from the textbook:

Properties of Vectors If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Here, $\vec{0} = \langle 0, 0, \dots, 0 \rangle$ and V_n is the set of all vectors in $\mathbb{R}^n$.

The Dot Product

The magnitude of a vector, $|\vec{v}|$, lets measure distances in $\mathbb{R}^n$.

What about the "direction" of $\vec{v}$?

The dot product is a tool used to

dot product is a tool used to measure the angle between two vectors.

$\boxed{\text{Def}}$ For two vectors $\vec{v} = \langle v_1, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, their **dot product** is the real number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n .$$

$\boxed{\text{Ex}}$ $\langle 2,4 \rangle \cdot \langle 3,-1 \rangle = 2 \cdot 3 + 4 \cdot {-1}$

$= 6 - 4 = 2$

$\langle -1,7,4 \rangle \cdot \langle 6,2,-\tfrac{1}{2} \rangle = -6 + 14 - 2 = 6$

Here are some properties of $\cdot$:

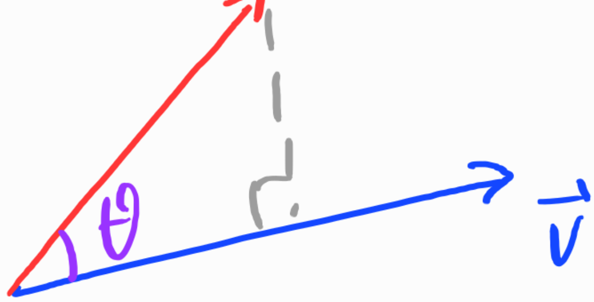
2 **Properties of the Dot Product** If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

For us, the dot product is most useful for computing angles.

$\boxed{\text{Theorem}}$ If θ is the angle between two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, then

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\theta).$$



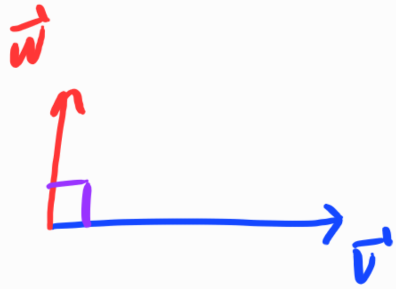
Ex The angle between $\vec{v} = \langle -1, 1, 2 \rangle$ and $\vec{w} = \langle 2, 1, -1 \rangle$ is

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)$$

$$= \arccos\left(\frac{-2+1-2}{\sqrt{1+1+4}\,\sqrt{4+1+1}}\right)$$

$$= \arccos\left(\frac{-3}{6}\right) = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}.$$

Def Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal if they meet at a right angle.



Notice that $\vec{v} \perp \vec{w} \iff \theta = \frac{\pi}{2}$

$$\iff \cos(\theta) = 0$$

$$\iff \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = 0$$

$$\iff \vec{v} \cdot \vec{w} = 0.$$

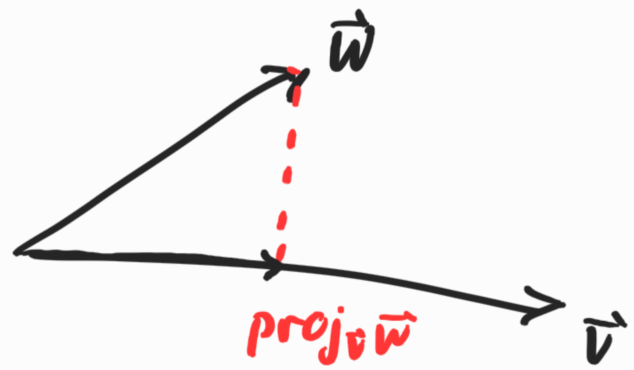
Ex If $\vec{v} = \langle 2, 2, -1 \rangle$ and $\vec{w} = \langle 5, -4, 2 \rangle$,

then $\vec{v} \cdot \vec{w} = 10 - 8 - 2 = 0.$

So $\vec{v}$ and $\vec{w}$ are orthogonal.

Def The **projection** of a vector $\vec{w}$ onto another vector $\vec{v}$ is the vector

$$\text{proj}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$



In plain English, $\text{proj}_{\vec{v}}\vec{w}$ is the "shadow" of $\vec{w}$ in the direction of $\vec{v}$.

$\boxed{\text{Ex}}$ The projection of $\vec{w} = \langle 2, 1, -1 \rangle$ onto $\vec{v} = \langle -1, 1, 2 \rangle$ is

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{-2+1-2}{1+1+4}\langle -1, 1, 2 \rangle$$

$$= \frac{-3}{6}\langle -1, 1, 2 \rangle = \left\langle \frac{1}{2}, \frac{-1}{2}, -1 \right\rangle.$$

Exercise 1: If $\vec{v}$ and $\vec{w}$ are orthogonal, what is $\mathrm{proj}_{\vec{v}}\,\vec{w}$? what if they are parallel?

The Cross Product

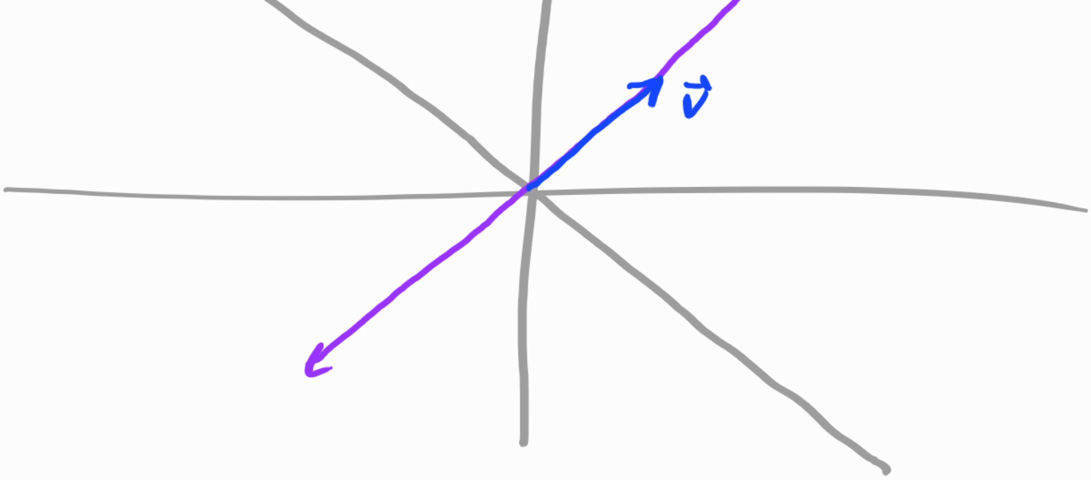
Let's focus our attention on $\mathbb{R}^3$.

Facts:

- A nonzero vector $\vec{v}$ specifies a line through the origin:

$$(x,y,z) = t\vec{v} = \langle tv_1, tv_2, tv_3 \rangle.$$

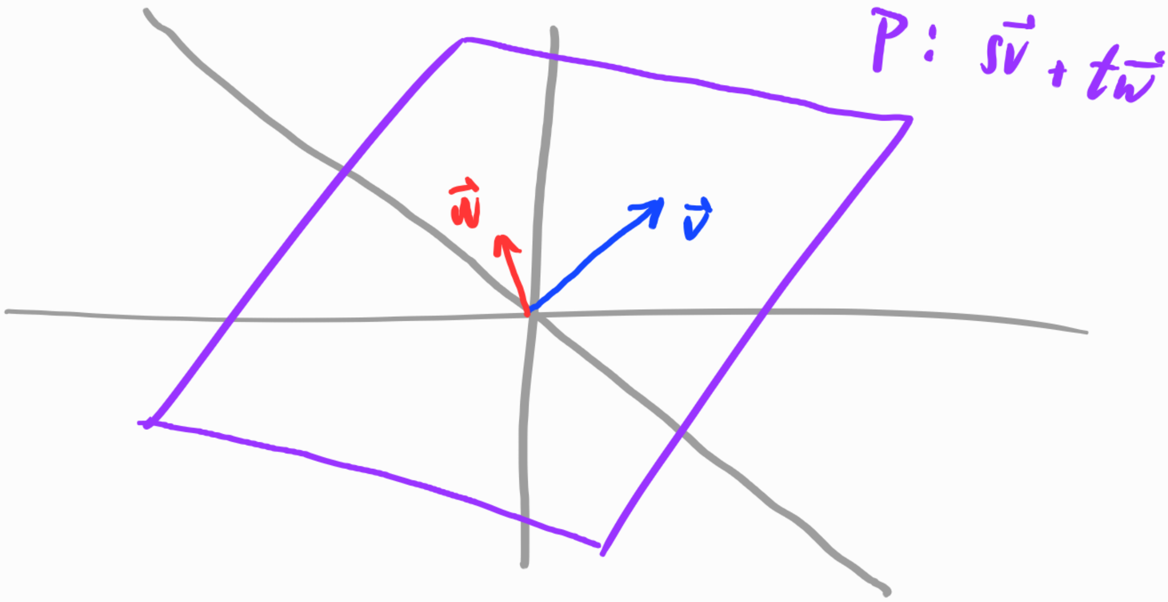
$L: \vec{x} = t\vec{v}$



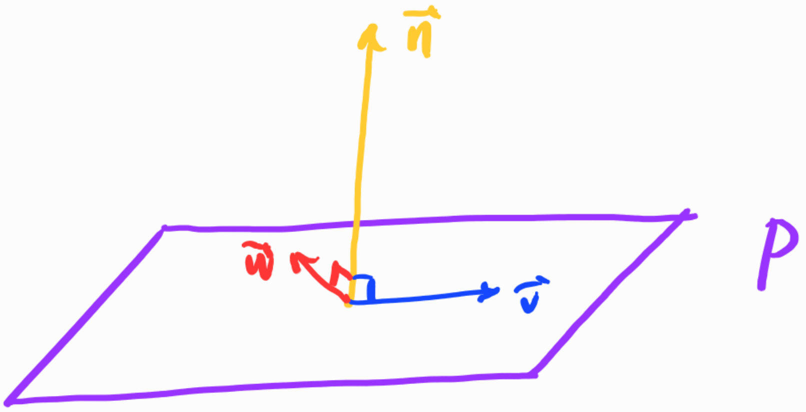
- Two nonparallel vectors $\vec{v}$ and $\vec{w}$ specify a plane through the origin:

$$(x, y, z) = s\vec{v} + t\vec{w}$$

$$= \langle sv_1 + tw_1,\ sv_2 + tw_2,\ sv_3 + tw_3 \rangle.$$



Alternatively, a plane through $\vec{0}$ can be specified by a normal vector which is orthogonal to every vector lying in the plane.



In coordinates, if $\vec{n} = \langle a, b, c \rangle$ is a normal vector for P, then every point in P is given by

$$\vec{n} \cdot \vec{x} = 0$$

$$n \cdot x = 0$$

or $\quad ax + by + cz = 0.$

The **cross product** lets us compute these normal vectors.

Def The **cross product** between two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ is the vector $\vec{v} \times \vec{w}$ with coordinates

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1 \rangle.$$

Equivalently, if $\quad i = \langle 1, 0, 0 \rangle$

$\qquad j = \langle 0, 1, 0 \rangle$

$$j = \langle 0, 1, 0 \rangle$$

$$k = \langle 0, 0, 1 \rangle$$

are the standard coordinate vectors in $\mathbb{R}^3$, then

$$\vec{v} \times \vec{w} = \begin{vmatrix} v_1 & w_1 & i \\ v_2 & w_2 & j \\ v_3 & w_3 & k \end{vmatrix}$$

$$= \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} k.$$

Theorem For any $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$,

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin(\theta).$$

In particular, $\vec{v} \times \vec{w}$ is orthogonal to

In particular, $\vec{v} \times \vec{w}$ is orthogonal to

both $\vec{v}$ and $\vec{w}$.

Exercise 2: Prove the Theorem.

$\boxed{\text{Ex}}$ Let's find a normal vector for the plane P spanned by $\vec{v} = \langle 1, 3, 4 \rangle$ and $\vec{w} = \langle 2, 7, -5 \rangle$.

We want a vector $\vec{n}$ which is orthogonal to every vector in P.

It's enough to find $\vec{n}$ orthogonal to $\vec{v}$ and $\vec{w}$. (why?)

By the Theorem, $\vec{n} = \vec{v} \times \vec{w}$ works:

$$\vec{v} \times \vec{w} = \langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle$$

$$= \begin{vmatrix} 1 & 2 & i \\ 3 & 7 & j \\ 4 & -5 & k \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 7 \\ 4 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 4 & -5 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} k$$

$$= (-15 - 28)\, i - (-5 - 8)\, j + (7 - 6)\, k$$

$$= \langle -43, 13, 1 \rangle.$$

Since $\vec{n} = \langle -43, 13, 1 \rangle$

Since $\vec{n} = \langle -43, 13, 1 \rangle$ is normal to P, every point in P can be written

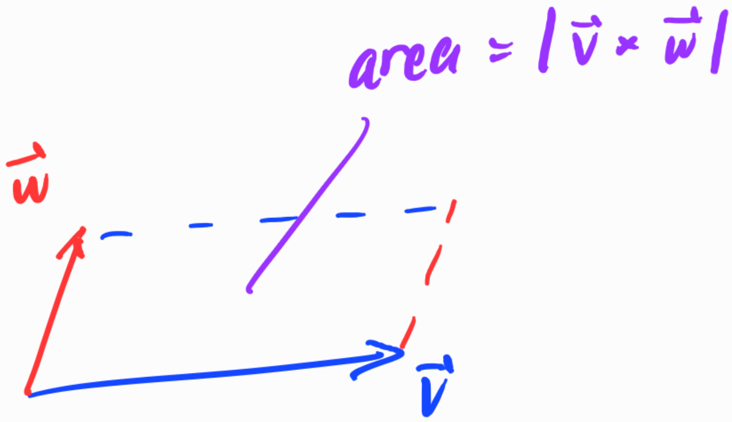
$$\vec{n} \cdot \vec{x} = 0$$

or $$-43x + 13y + z = 0$$

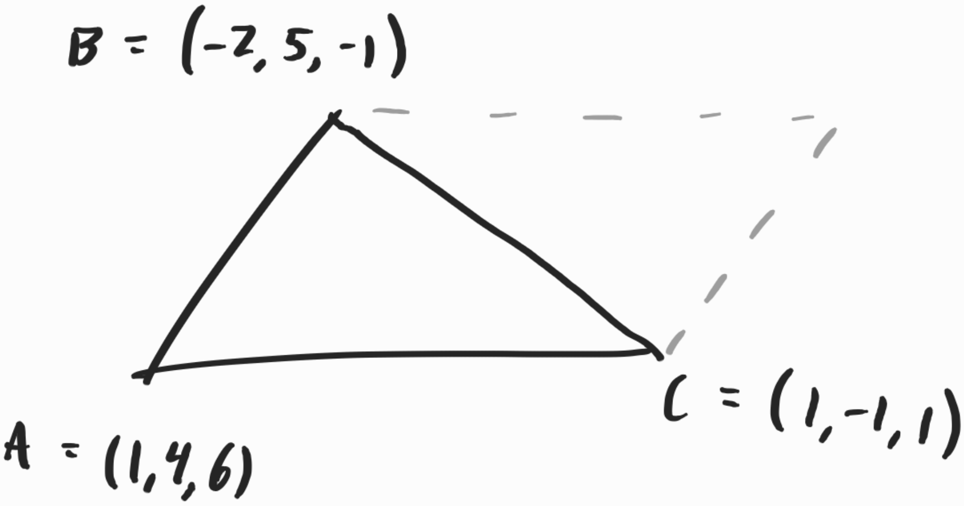
or $$z = 43x - 13y.$$

$\boxed{\text{Theorem}}$ For two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, $|\vec{v} \times \vec{w}|$ is equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$

$\vec{v}$ and $\vec{w}$.



$\boxed{\text{Ex}}$ Let's use the cross product to find the area of the triangle



One way to do this is to form a parallelogram with initial sides

$$\vec{v} = \overrightarrow{AB} = \langle -3, 1, -7 \rangle$$

and $\vec{w} = \overrightarrow{AC} = \langle 0, -5, -5 \rangle$.

Then the area of the parallelogram is

$|\vec{v} \times \vec{w}|$ and

$$\vec{v} \times \vec{w} = \begin{vmatrix} -3 & 0 & i \\ 1 & -5 & j \\ -7 & -5 & k \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -5 \\ -7 & -5 \end{vmatrix} i - \begin{vmatrix} -3 & 0 \\ -7 & -5 \end{vmatrix} j + \begin{vmatrix} -3 & 0 \\ 1 & -5 \end{vmatrix} k$$

$$= (-5 - 35)i - (15 - 0)j + (15 - 0)k$$

$$= \langle -40, -15, 15 \rangle.$$

Then

$$▱ = \left| \langle -40, -15, 15 \rangle \right|$$

$$= \sqrt{(-40)^2 + (-15)^2 + 15^2}$$

$$= \sqrt{1600 + 225 + 225}$$

$$= \sqrt{2050}$$

So $\triangle = \frac{1}{2}\, ▱ = \frac{1}{2}\sqrt{2050}$.

$$\left(= \frac{5}{2}\sqrt{82} \right)$$

Next time: more on lines and planes.