$\boxed{\text{Lecture 12.4}}$

Last time:

- Vectors are
  - a direction and a magnitude
  - an arrow in $\mathbb{R}^n$
  - a unique arrow starting at the origin in $\mathbb{R}^n$
  - a list of coordinates.

- The dot product of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n .$$

- If θ is the angle between $\vec{v}$ and $\vec{w}$, then

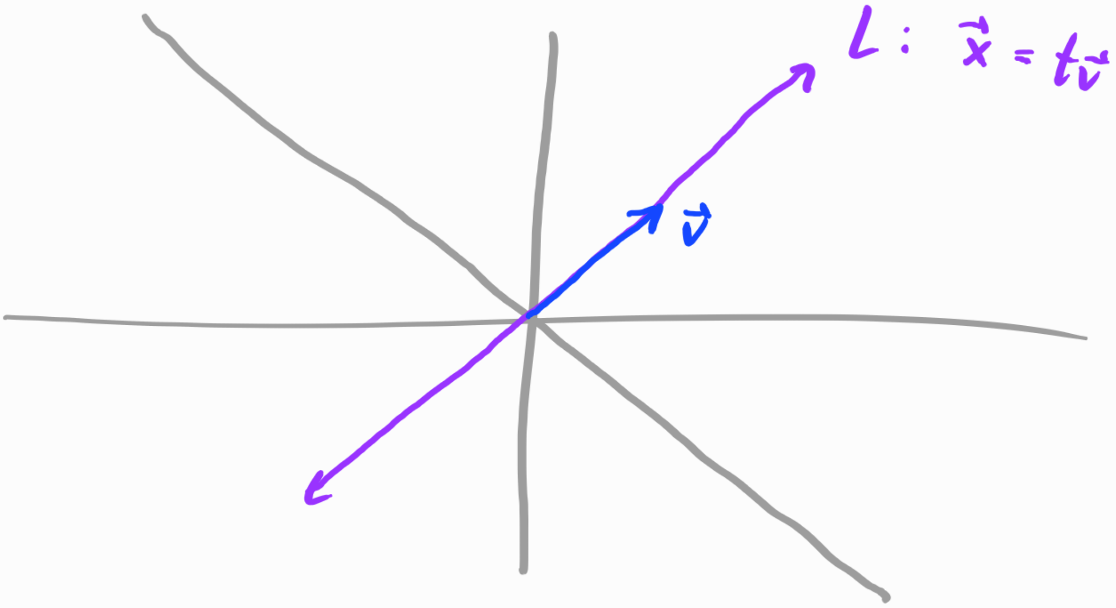
$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta).$$

The Cross Product

Let's focus our attention on $\mathbb{R}^3$.

Facts: • A nonzero vector $\vec{v}$ specifies a line through the origin:

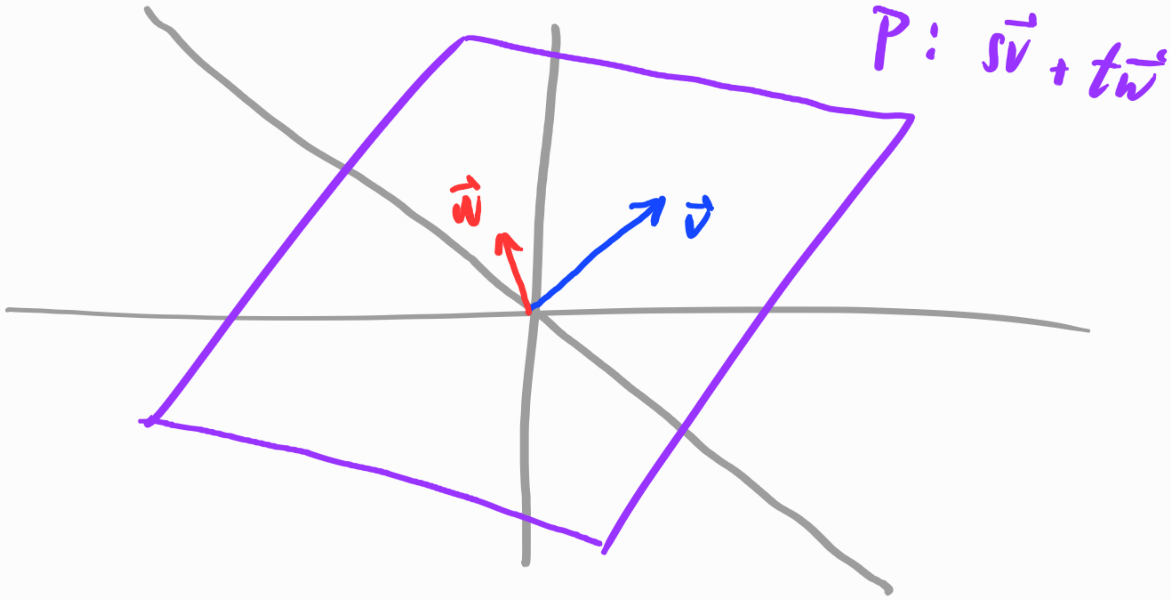
$$(x, y, z) = t\vec{v} = \langle tv_1, tv_2, tv_3 \rangle.$$



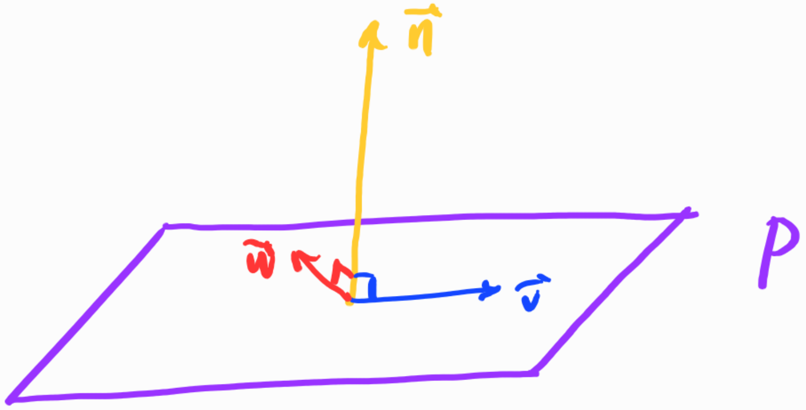
- Two nonparallel vectors $\vec{v}$ and $\vec{w}$ specify a plane through the origin:

$$(x, y, z) = s\vec{v} + t\vec{w}$$

$$= \langle sv_1 + tw_1,\ sv_2 + tw_2,\ sv_3 + tw_3 \rangle.$$



Alternatively, a plane through $\vec{0}$ can be specified by a normal vector which is orthogonal to every vector lying in the plane.



In coordinates, if $\vec{n} = \langle a, b, c \rangle$ is a normal vector for P, then every point in P is given by

$$\vec{n} \cdot \vec{x} = 0$$

or $\quad ax + by + cz = 0.$

The cross product lets us compute

these normal vectors.

$\boxed{\text{Def}}$ The **cross product** between two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ is the vector $\vec{v} \times \vec{w}$ with coordinates

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1 \rangle.$$

Equivalently, if

$$i = \langle 1, 0, 0 \rangle$$
$$j = \langle 0, 1, 0 \rangle$$
$$k = \langle 0, 0, 1 \rangle$$

are the standard coordinate vectors in $\mathbb{R}^3$ then

$\mathbb{R}$, the

$$\vec{v} \times \vec{w} = \begin{vmatrix} v_1 & w_1 & i \\ v_2 & w_2 & j \\ v_3 & w_3 & k \end{vmatrix}$$

$$= \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} k.$$

Theorem For any $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$,

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin(\theta).$$

In particular, $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Exercise 1: Prove the Theo

Exercise [illegible]: Prove the Theorem.

$\boxed{\text{Ex}}$ Let's find a normal vector for the plane P spanned by $\vec{v} = \langle 1, 3, 4 \rangle$ and $\vec{w} = \langle 2, 7, -5 \rangle$.

We want a vector $\vec{n}$ which is orthogonal to every vector in P.

It's enough to find $\vec{n}$ orthogonal to $\vec{v}$ and $\vec{w}$. (why?)

By the Theorem, $\vec{n} = \vec{v} \times \vec{w}$ works:

$$\vec{v} \times \vec{w} = \langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle$$

$$= \begin{vmatrix} 1 & 2 & i \\ 3 & 7 & j \\ 4 & -5 & k \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 7 \\ 4 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 4 & -5 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} k$$

$$= (-15-28)i - (-5-8)j + (7-6)k$$

$$= \langle -43, 13, 1 \rangle.$$

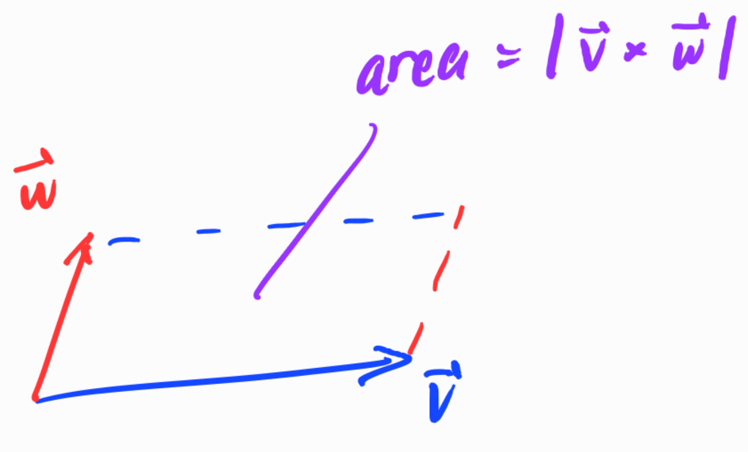
Since $\vec{n} = \langle -43, 13, 1 \rangle$ is normal to P, every point in P can be written

$$\vec{n} \cdot \vec{x} = 0$$

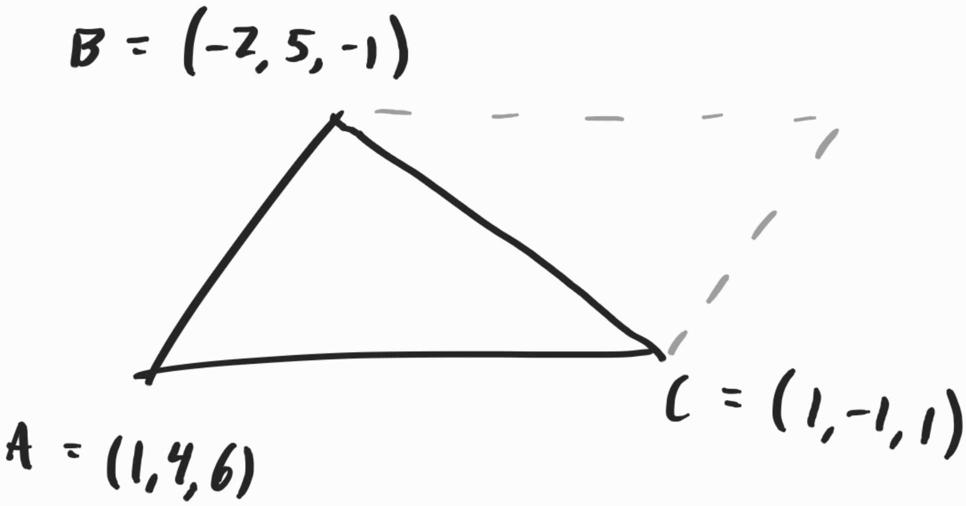
or $\quad -43x + 13y + z = 0$

or $\quad z = 43x - 13y.$

Theorem For two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, $|\vec{v} \times \vec{w}|$ is equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$.



Ex Let's use the cross product to find the area of the triangle



One way to do this is to form a parallelogram with initial sides

$$\vec{v} = \overrightarrow{AB} = \langle -3, 1, -7 \rangle$$

and $\vec{w} = \overrightarrow{AC} = \langle 0, -5, -5 \rangle$.

Then the area of the parallelogram is

$|\vec{v} \times \vec{w}|$ and

$$\vec{v} \times \vec{w} = \begin{vmatrix} -3 & 0 & i \\ 1 & -5 & j \\ -7 & -5 & k \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -5 \\ -7 & -5 \end{vmatrix} i - \begin{vmatrix} -3 & 0 \\ -7 & -5 \end{vmatrix} j + \begin{vmatrix} -3 & 0 \\ 1 & -5 \end{vmatrix} k$$

$$= (-5 - 35) i - (15 - 0) j + (15 - 0) k$$

$$= \langle -40, -15, 15 \rangle.$$

Then

$$\text{▱} = |\langle -40, -15, 15 \rangle|$$

$$= \sqrt{(-40)^2 + (-15)^2 + 15^2}$$

$$= \sqrt{1600 + 225 + 225}$$

$$= \sqrt{2050}$$

So $\triangle = \frac{1}{2}\,▱ = \frac{1}{2}\sqrt{2050}.$

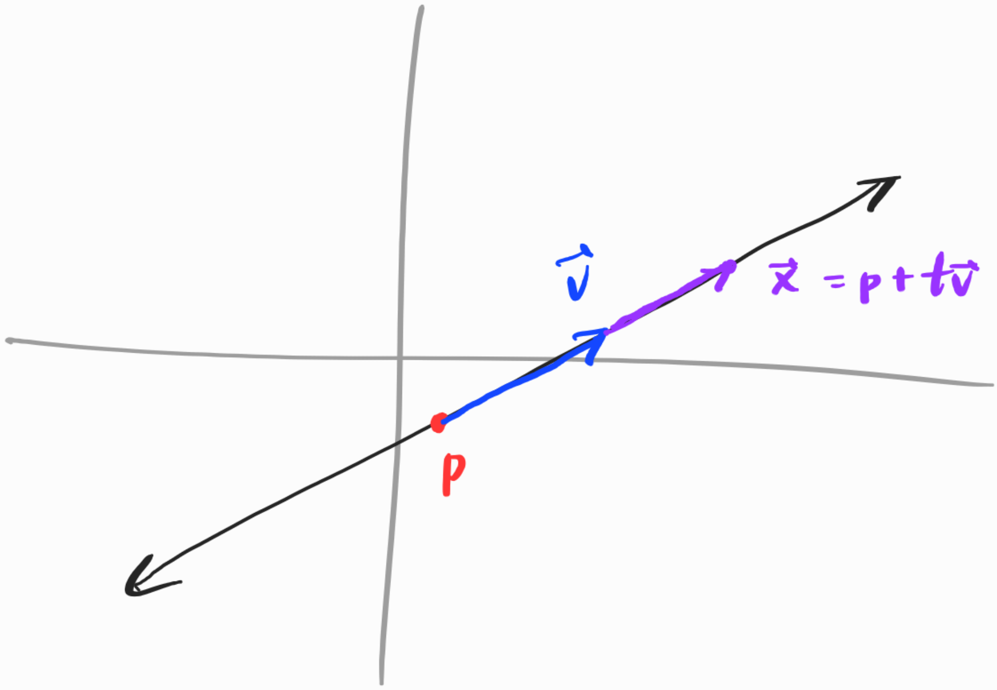
$$\left(= \frac{5}{2}\sqrt{82} \right)$$

$\boxed{\text{Lines and Planes}}$

A line in $\mathbb{R}^n$ is determined by a point $P = (p_1, \dots, p_n)$ and a slope, encoded by a vector $\vec{v} = \langle v_1, \dots, v_n \rangle$. Every point $\vec{x} = (x_1, \dots, x_n)$ on this line satisfies the equation

$$\vec{x} = P + t\vec{v} \quad \text{for some } t \text{ in } \mathbb{R}$$

OR $(x_1, \ldots, x_n) = (p_1 + tv_1, \ldots, p_n + tv_n)$



$\boxed{\text{Ex}}$ An equation for the line L in $\mathbb{R}^3$ passing through $(5,1,3)$ and parallel to $\vec{v} = \langle 1, 4, -2 \rangle$ is

$$\vec{x} = (5,1,3) + t\langle 1,4,-2 \rangle$$

$$= (5+t,\ 1+4t,\ 3-2t)$$

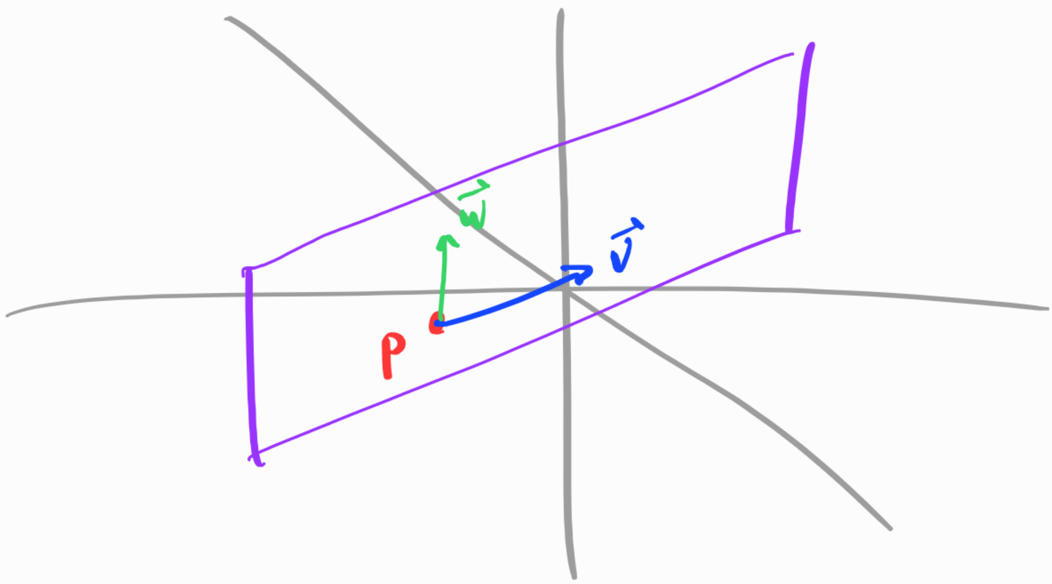
OR $x = 5+t, \quad y = 1+4t, \quad z = 3-2t.$

parametric equations

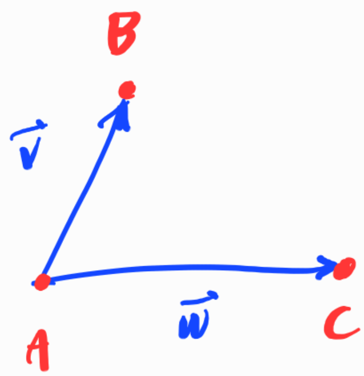
Next, a plane in $\mathbb{R}^n$ is specified by a point P and two nonparallel vectors $\vec{v}$ and $\vec{w}$.

Every point $\vec{x}$ on the plane satisfies

$$\vec{x} = P + s\vec{v} + t\vec{w}.$$



$\boxed{\text{Ex}}$ Let P be the plane passing through $A = (1,2,2)$, $B = (-1,4,1)$ and $C = (4,5,6)$ in $\mathbb{R}^3$.



Two vectors in P are

$$\vec{v} = \overrightarrow{AB} = \langle -2, 2, -1 \rangle$$

and $\vec{w} = \overrightarrow{AC} = \langle 3, 3, 4 \rangle$.

Then a parametric equation for P is

$$\vec{x} = A + s\vec{v} + t\vec{w}$$

$$= (1,2,2) + s\langle -2, 2, -1 \rangle + t\langle 3, 3, 4 \rangle$$

$$(1,2,2) + s\langle -2,2,-1\rangle + t\langle 3,3,4\rangle$$

OR
$$\begin{aligned} x &= 1 - 2s + 3t \\ y &= 2 + 2s + 3t \\ z &= 2 - s + 4t. \end{aligned}$$

In $\mathbb{R}^3$, a plane can also be specified by a point P and a normal vector $\vec{n}$:

$$\vec{n} \cdot (\vec{x} - P) = \vec{0}.$$

$\boxed{\text{Ex}}$ In the example above, we can take $\vec{n} = \vec{v} \times \vec{w} = \langle 11, 5, -12\rangle$ and $P = A = (1,2,2)$ to get

$$\langle 11, 5, -12 \rangle \cdot \langle x-1, y-2, z-2 \rangle = 0$$

OR $$11(x-1) + 5(y-2) - 12(z-2) = 0$$

OR $$11x + 5y - 12z = -3.$$

Exercise 2: Check that every parametric solution satisfies the normal equation.

$\boxed{\text{Ex}}$ What is the angle of incidence between the planes

$$x + y + z = 1 \quad \text{and} \quad x - 2y + 3z = 1 \text{ ?}$$

These planes have normal vecters

$\vec{n}_1 = \langle 1, 1, 1 \rangle$ and $\vec{n}_2 = \langle 1, -2, 3 \rangle$

respectively, so if θ is the angle between

them,

$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \, |\vec{n}_2|} = \frac{1 - 2 + 3}{\sqrt{1+1+1}\,\sqrt{1+4+9}}$$

$$= \frac{2}{\sqrt{42}}$$

$$\Longrightarrow \quad \theta = \arccos\left(\frac{2}{\sqrt{42}}\right) \approx 1.26 \text{ or } 72.02°.$$

$\boxed{\text{Theorem}}$ Let $P = (x_0, y_0, z_0)$ be a point and

$ax + by + cz + d = 0$ be a plane in $\mathbb{R}^3$.

Then the distance from P to the

plane is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

or $\frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$ where $\vec{n} = \langle a, b, c \rangle$ is the given normal vector to the plane and $\vec{b} = \overrightarrow{PQ}$ for any Q in the plane.

$\boxed{\text{Ex}}$ The distance between the planes

$$10x + 2y - 2z = 5 \quad \text{and} \quad 5x + y - z = 1$$

is positive because the planes have parallel normal vectors and are therefore parallel themselves:

$$\vec{n}_1 = \langle 10, 2, -2 \rangle = 2\vec{n}_2$$

$\vec{n}_2 = \langle 5, 1, -1 \rangle.$

Pick any point on the first plane, say

$P = (\frac{1}{2}, 0, 0)$, and compute:

$$\text{dist} = \frac{|\vec{n}_2 \cdot \vec{b}|}{|\vec{n}_2|} = \frac{|5 \cdot \frac{1}{2} + 1 \cdot 0 - 1 \cdot 0 - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}}$$

$$= \frac{3/2}{\sqrt{27}} = \frac{1}{2\sqrt{3}}.$$

Next time: multivariable functions.