Lecture 12.5/14.1

Last time:

- The cross product of $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ is the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1 \rangle.$$

- For any $\vec{v}, \vec{w}$, $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$.

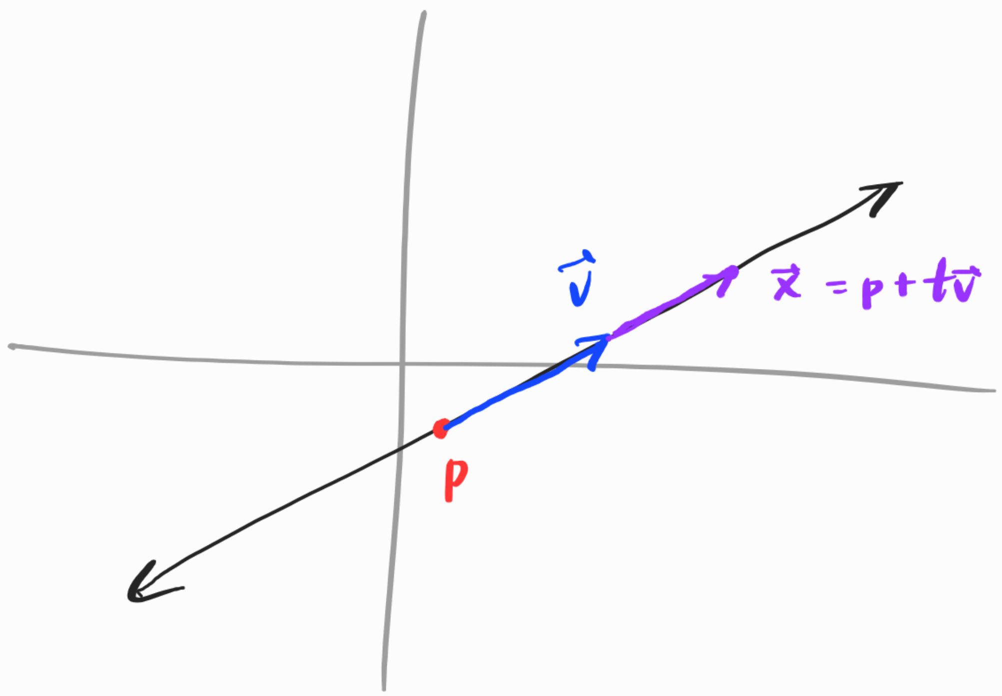
- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

Lines and Planes

A line in $\mathbb{R}^n$ is determined by a point $P = (p_1, \dots, p_n)$ and a slope, encoded by a vector $\vec{v} = \langle v_1, \dots, v_n \rangle$. Every point $\vec{x} = (x_1, \dots, x_n)$ on this line satisfies the equation

$$\vec{x} = P + t\vec{v} \quad \text{for some } t \text{ in } \mathbb{R}$$

OR $$(x_1, \dots, x_n) = (p_1 + tv_1, \dots, p_n + tv_n)$$



$\boxed{\text{Ex}}$ An equation for the line L in $\mathbb{R}^3$ passing through $(5,1,3)$ and parallel to $\vec{v} = \langle 1, 4, -2 \rangle$ is

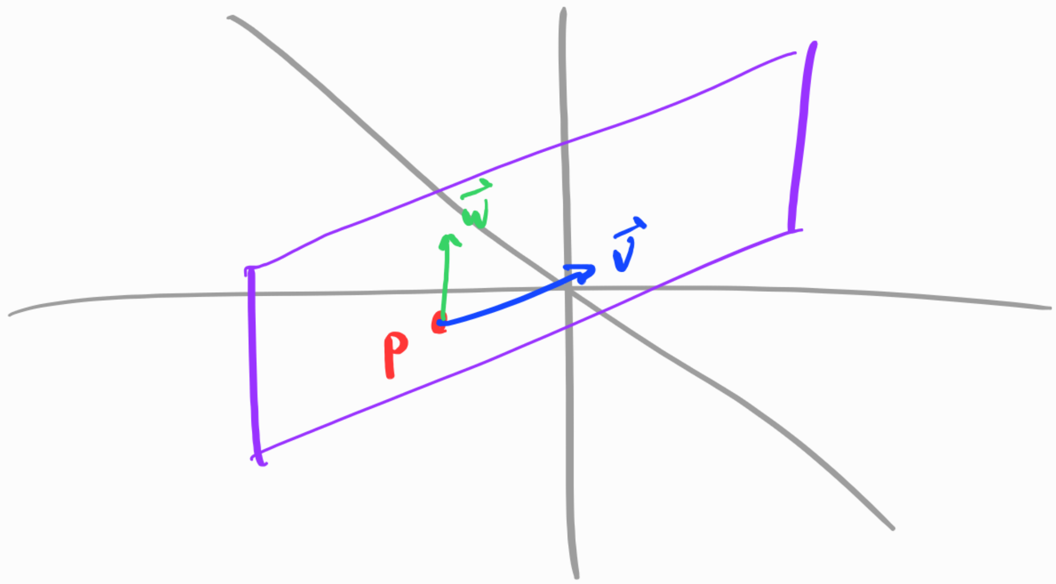
$$\vec{x} = (5,1,3) + t\langle 1,4,-2 \rangle$$
$$= (5+t,\ 1+4t,\ 3-2t)$$

OR $\underbrace{x = 5+t, \quad y = 1+4t, \quad z = 3-2t.}_{\text{parametric equations}}$

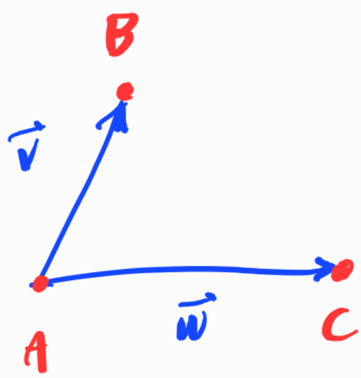
Next, a plane in $\mathbb{R}^n$ is specified by a point P and two nonparallel vectors $\vec{v}$ and $\vec{w}$.

Every point $\vec{x}$ on the plane satisfies

$$\vec{x} = P + s\vec{v} + t\vec{w}.$$



$\boxed{\text{Ex}}$ Let P be the plane passing through $A = (1, 2, 2)$, $B = (-1, 4, 1)$ and $C = (4, 5, 6)$ in $\mathbb{R}^3$.



Two vectors in P are

$$\vec{v} = \vec{AB} = \langle -2, 2, -1 \rangle$$

and $\vec{w} = \vec{AC} = \langle 3, 3, 4 \rangle$.

Then a parametric equation for P is

$$\vec{x} = A + s\vec{v} + t\vec{w}$$

$$= (1,2,2) + s\langle -2, 2, -1 \rangle + t\langle 3,3,4 \rangle$$

OR
$$x = 1 - 2s + 3t$$
$$y = 2 + 2s + 3t$$
$$z = 2 - s + 4t.$$

In $\mathbb{R}^3$ a plane can also be

In $\mathbb{R}$, a plane can also be specified by a point P and a normal vector $\vec{n}$:

$$\vec{n} \cdot (\vec{x} - P) = \vec{0}.$$

$\boxed{\text{Ex}}$ In the example above, we can take $\vec{n} = \vec{v} \times \vec{w} = \langle 11, 5, -12 \rangle$ and $P = A = (1, 2, 2)$ to get

$$\langle 11, 5, -12 \rangle \cdot \langle x-1, y-2, z-2 \rangle = 0$$

OR $$11(x-1) + 5(y-2) - 12(z-2) = 0$$

OR $$11x + 5y - 12z = -3.$$

Exercise 1: Check that every parametric solution satisfies the normal equation.

$\boxed{\text{Ex}}$ What is the angle of incidence between the planes

$$x + y + z = 1 \quad \text{and} \quad x - 2y + 3z = 1 \ ?$$

These planes have normal vecters

$$\vec{n}_1 = \langle 1, 1, 1 \rangle \quad \text{and} \quad \vec{n}_2 = \langle 1, -2, 3 \rangle$$

respectively, so if θ is the angle between them,

$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{1 - 2 + 3}{\sqrt{1+1+1}\,\sqrt{1+4+9}}$$

$$2$$

$$= \frac{\quad}{\sqrt{42}}$$

$$\Rightarrow \quad \theta = \arccos\left(\frac{2}{\sqrt{42}}\right) \approx 1.26 \text{ or } 72.02°.$$

$\boxed{\text{Theorem}}$ Let $P = (x_0, y_0, z_0)$ be a point and $ax + by + cz + d = 0$ be a plane in $\mathbb{R}^3$. Then the distance from P to the plane is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

or $\dfrac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$ where $\vec{n} = \langle a, b, c \rangle$ is the given normal vector to the plane and $\vec{b} = \overrightarrow{PQ}$ for any Q in the plane.

$\boxed{\text{Ex}}$ The distance between the planes

$$10x + 2y - 2z = 5 \quad \text{and} \quad 5x + y - z = 1$$

is positive because the planes have parallel normal vectors and are therefore parallel themselves:

$$\vec{n}_1 = \langle 10, 2, -2 \rangle = 2\vec{n}_2$$

$$\vec{n}_2 = \langle 5, 1, -1 \rangle.$$

Pick any point on the first plane, say $P = (\frac{1}{2}, 0, 0)$, and compute:

$$\text{dist} = \frac{|\vec{n}_2 \cdot \vec{b}|}{|\vec{n}_2|} = \frac{|5 \cdot \frac{1}{2} + 1 \cdot 0 - 1 \cdot 0 - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}}$$

$$= \frac{3/2}{\sqrt{27}} = \frac{1}{2\sqrt{3}}.$$

Multivariable Functions

For a multivariable function $f(x_1, \dots, x_n)$, its **domain** is the set

$$D = \left\{ (x_1, \dots, x_n) \text{ in } \mathbb{R}^n \;\middle|\; \begin{array}{c} f(x_1, \dots, x_n) \text{ is} \\ \text{defined} \end{array} \right\}.$$

The main things to check for with the domain are the same as in single variable calculus:

- denominators should be nonzero

$\boxed{\text{Ex}}$ $f(x,y) = \dfrac{1}{x-y^2} \Rightarrow x \neq y^2$

- arguments of square roots (or any even power root) should be nonnegative

$\boxed{\text{Ex}}$ $f(x,y,z) = \sqrt{64-x^2-y^2-z^2}$

$\Rightarrow \quad 64-x^2-y^2-z^2 \geq 0$

OR $\quad x^2+y^2+z^2 \leq 64$

- ln has to input positive numbers

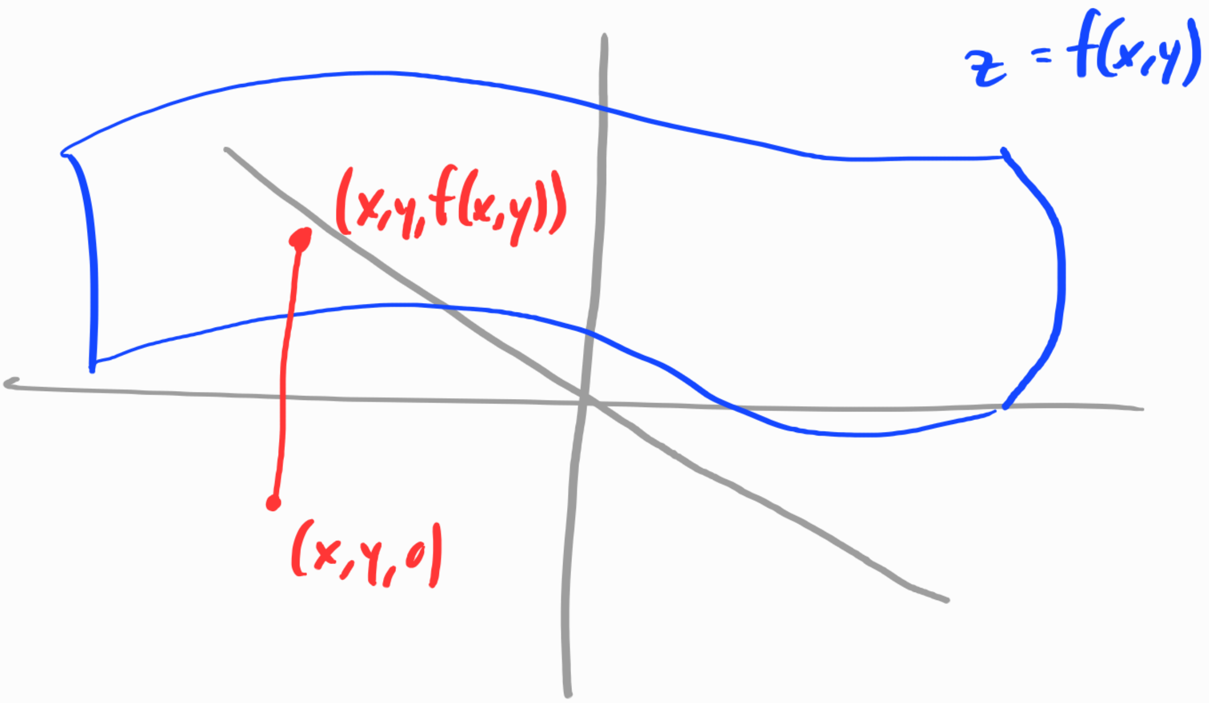
$\boxed{\text{Ex}}$ $f(x,y) = \ln(5x-10y)$

$\Rightarrow \quad 5x-10y > 0$

OR $\quad x > 2y$

- other functions with domain restrictions, e.g. $\arcsin(-)$

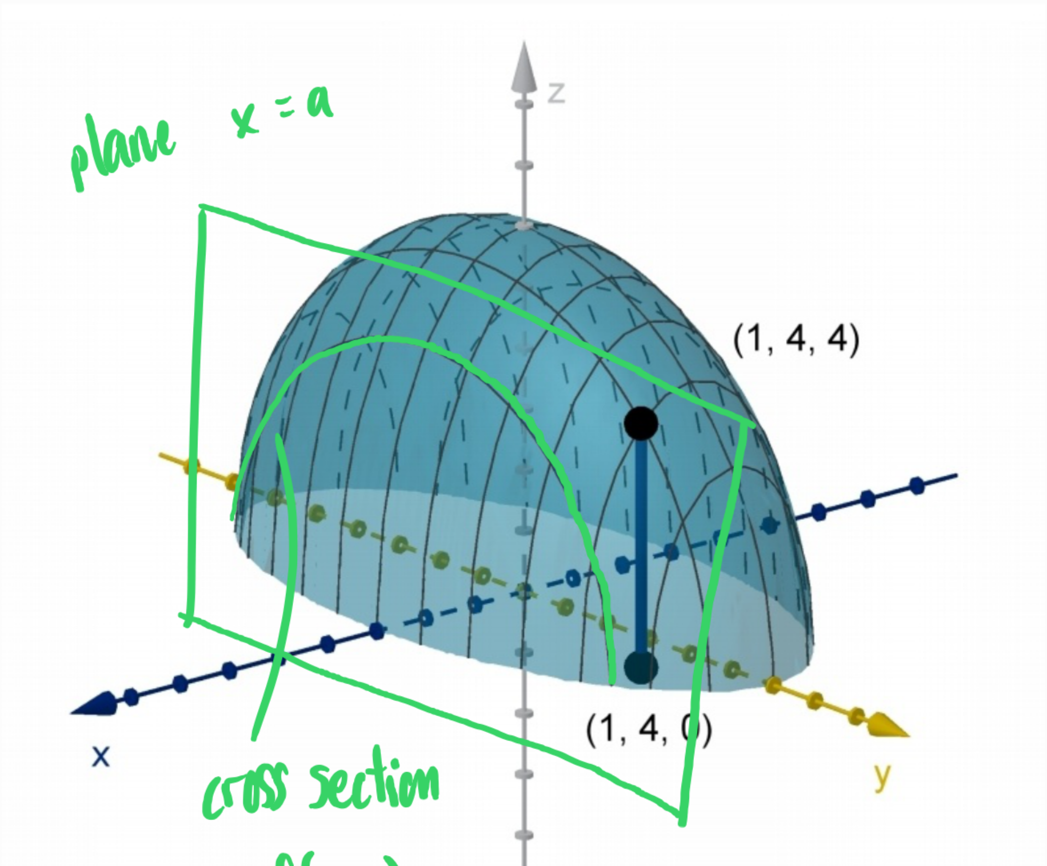
For the most part, we'll focus on functions $f(x,y)$ with domains in $\mathbb{R}^2$ and graphs in $\mathbb{R}^3$:



A cross section of a graph

$$z = f(x,y)$$

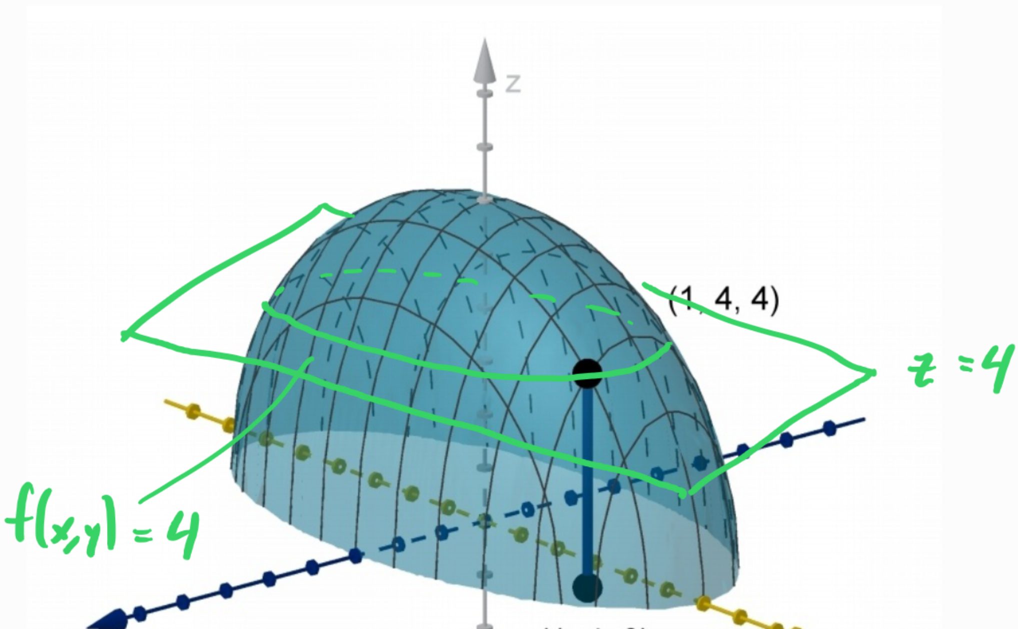
is the set of points lying on the intersection of the graph with any plane (in $\mathbb{R}^3$, given by $ax + by + cz + d = 0$).

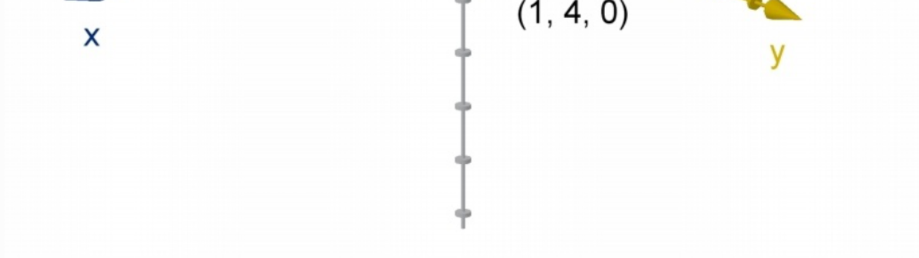


The most important type of cross section for us is a level curve, which is of the form

$$f(x,y) = k \quad \text{for some } k$$

i.e. a cross section cut out by a horizontal plane $z = k$.





By sketching multiple level curves of f in the xy-plane, we can start to visualize the graph of f topographically.

$\boxed{\text{Ex}}$ Here are some level curves of

$$f(x,y) = 6 - 3x - 2y,$$

for $k = -6, 0, 6, 12$:

<u>$k = -6$</u> :

$$6 - 3x - 2y = -6$$

$$\Longrightarrow \quad 2y = -3x + 12$$

$$\Longrightarrow \quad y = -\frac{3}{2}x + 6$$

$\underline{k=0}$: $6 - 3x - 2y = 0$

$\Rightarrow 2y = -3x + 6$

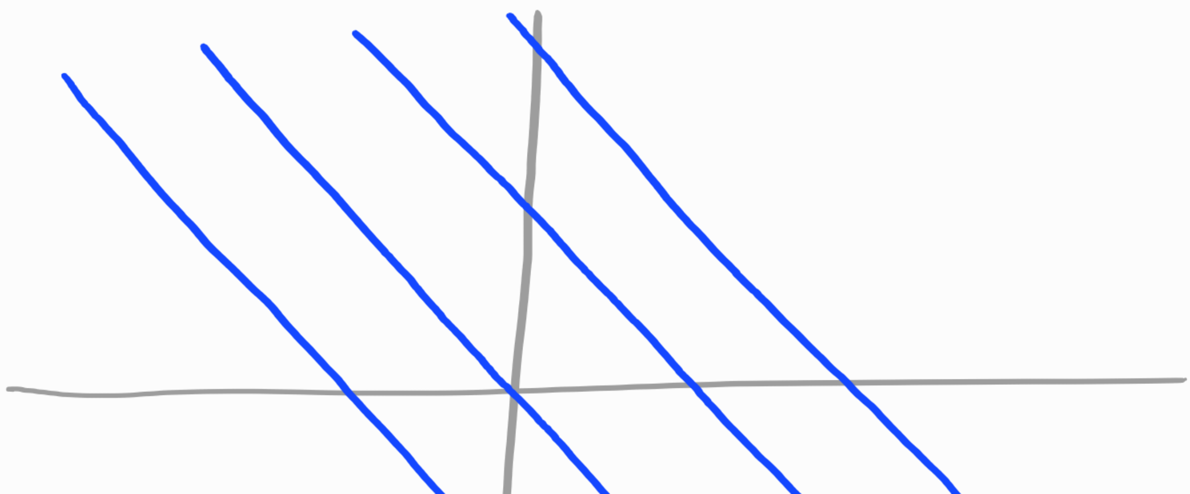
$\Rightarrow y = -\frac{3}{2}x + 3$

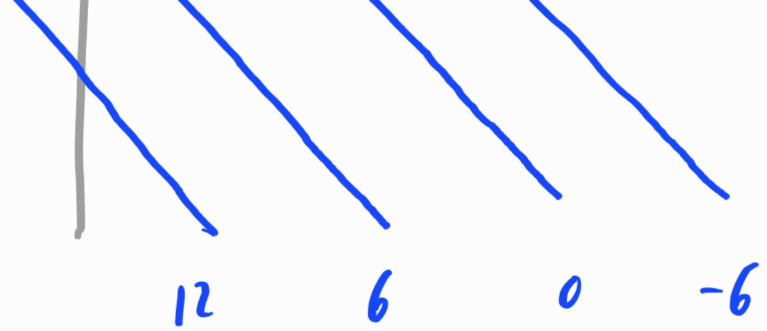
$\underline{k=6}$: $6 - 3x - 2y = 6$

$\Rightarrow y = -\frac{3}{2}x$

$\underline{k=12}$: $6 - 3x - 2y = 12$

$\Rightarrow y = -\frac{3}{2}x - 3$



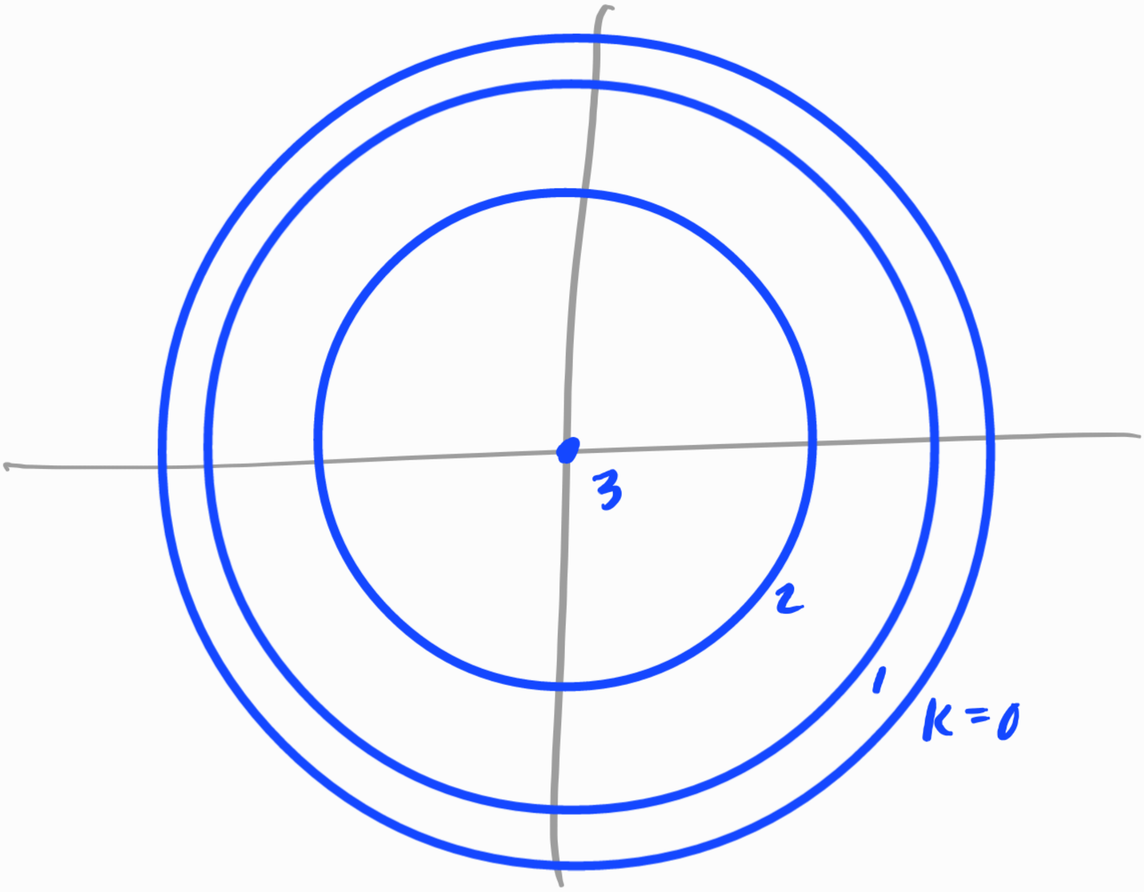


$\boxed{\text{Ex}}$ It's usually straightforward to find all level curves algebraically and then graph them. For example, if $f(x,y) = \sqrt{9-x^2-y^2}$ the level curves are all concentric circles!

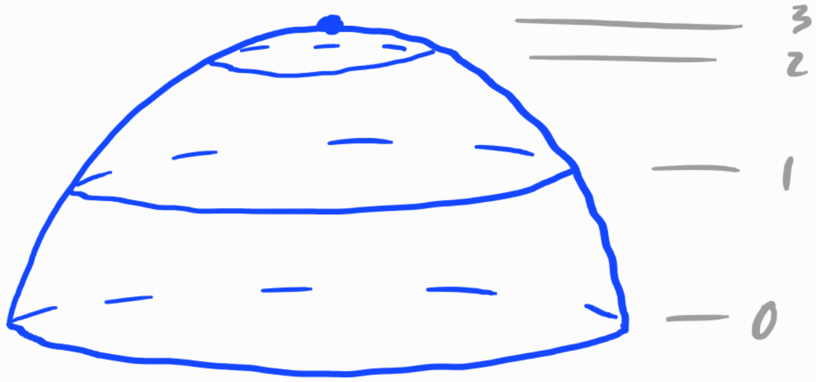
$$\sqrt{9-x^2-y^2} = k$$

$$\Longrightarrow \quad 9-x^2-y^2 = k^2$$

$$\Longrightarrow \quad x^2+y^2 = 9-k^2.$$



To visualize this in $\mathbb{R}^3$, keep in mind the k-values are heights:



Next time: limits.