$\boxed{\text{Lecture 13.1}}$

Last time:

- The surface area of a surface S defined by a graph $z = f(x,y)$ over a region R in the xy-plane is

$$A(S) = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Vector Valued Functions

So far, we've studied the calculus of functions of the form $f(x_1, \dots, x_n)$, with multiple inputs but only one output.

Why not allow multiple inputs and outputs?

$\boxed{\text{Def}}$ A vector-valued function is a function

$$f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$$

for input variables $x_1, \dots, x_n$ and "component function" $f_1, \dots, f_m$ depending on those inputs.

$\boxed{\text{Ex}}$ When $m = n = 1$, we get a single variable function $f(x)$.

$\boxed{\text{Ex}}$ When $m = 1$, we get one multivariable function $f(x_1, \dots, x_n)$.

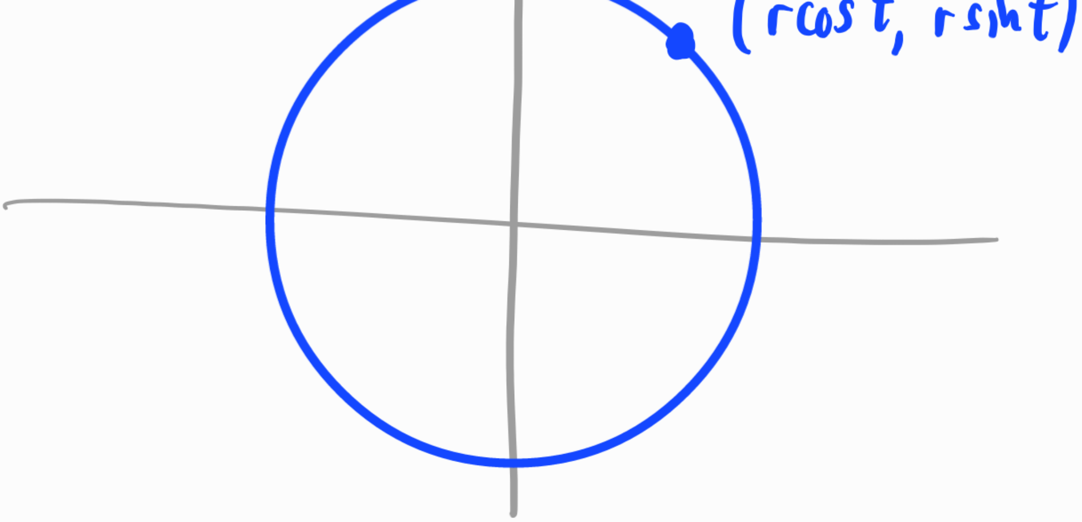
$\boxed{\text{Ex}}$ When $n = 1$, we write $t = x_1$ and say

$$f(t) = \langle f_1(t), \dots, f_m(t) \rangle$$

is a curve in $\mathbb{R}^m$. For example,

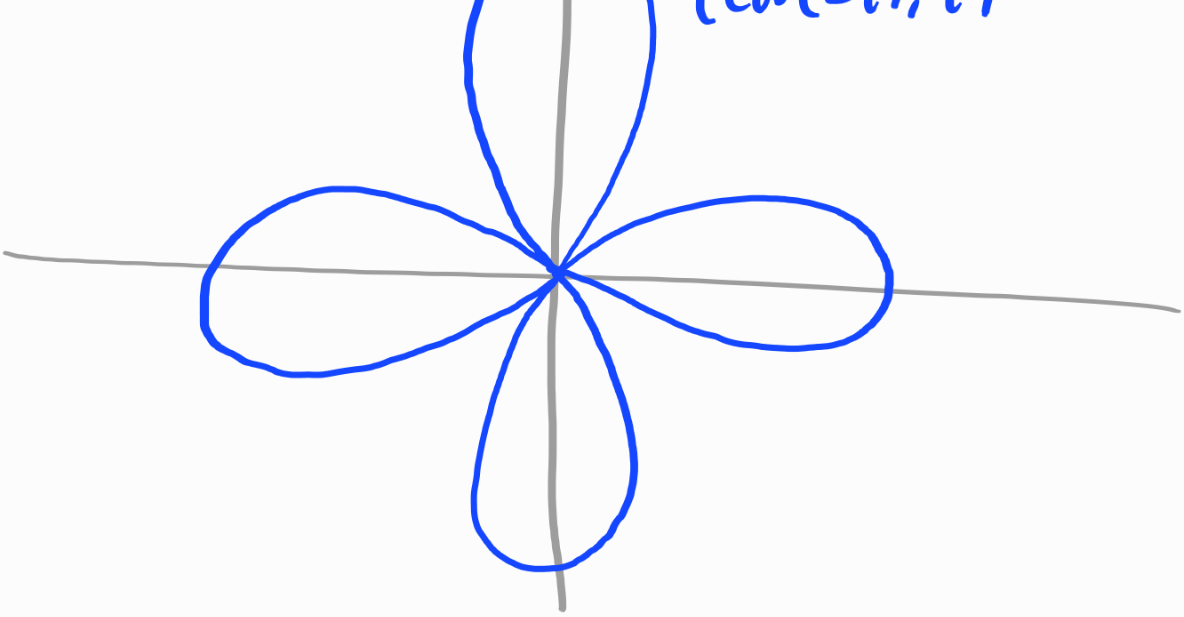
$$f(t) = \langle r\cos t, r\sin t \rangle$$

traces out the circle of radius r centered at the origin in $\mathbb{R}^2$:



Such a curve is often called parametric, with parameter t.

Ex The 4-petaled flower from lecture 15.3 is the curve $f(t) = \langle \cos(2t), t \rangle$ in polar coordinates:



Ex Any graph $y = f(x)$ can be written as a parametric curve

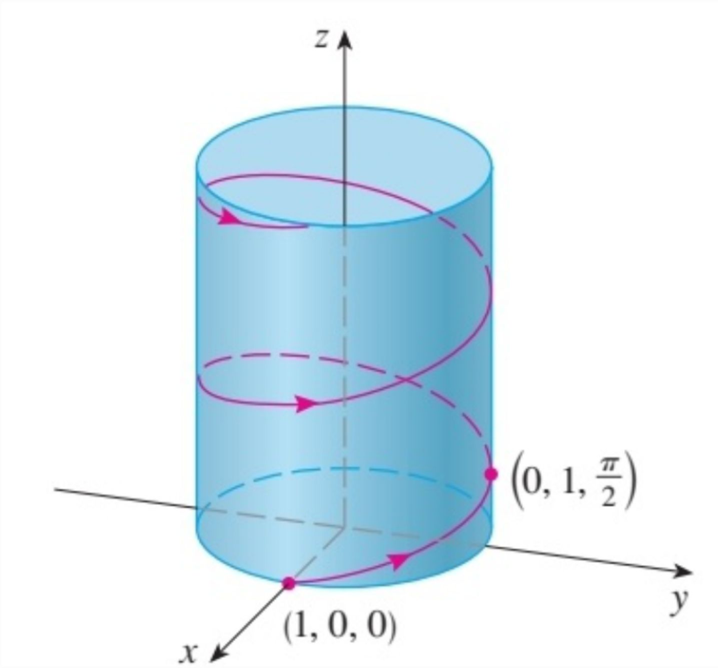
$$F(t) = \langle t, f(t) \rangle.$$

On the other hand, not every parametric curve $\langle f_1(t), f_2(t) \rangle$ has an "explicit" form relating the x- and y-coordinates.

$\boxed{Ex}$ The curve

$$r(t) = \langle \cos t, \sin t, t \rangle$$

in $\mathbb{R}^3$ looks like



$\boxed{Ex}$ A line in $\mathbb{R}^n$ can be written

$$\vec{x} = p + t\vec{v}$$

where p is a point on the line and $\vec{v}$ is a vector parallel to the line.

This can be viewed as a curve:

$$L(t) = \langle p_1 + tv_1, \dots, p_n + tv_n \rangle$$

where $p = (p_1, \dots, p_n)$, $\vec{v} = \langle v_1, \dots, v_n \rangle$.

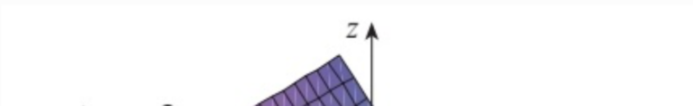
Exercise 1: Write a parametric equation for **(a)** the circle in $\mathbb{R}^2$ with center $(1,3)$ and radius 5

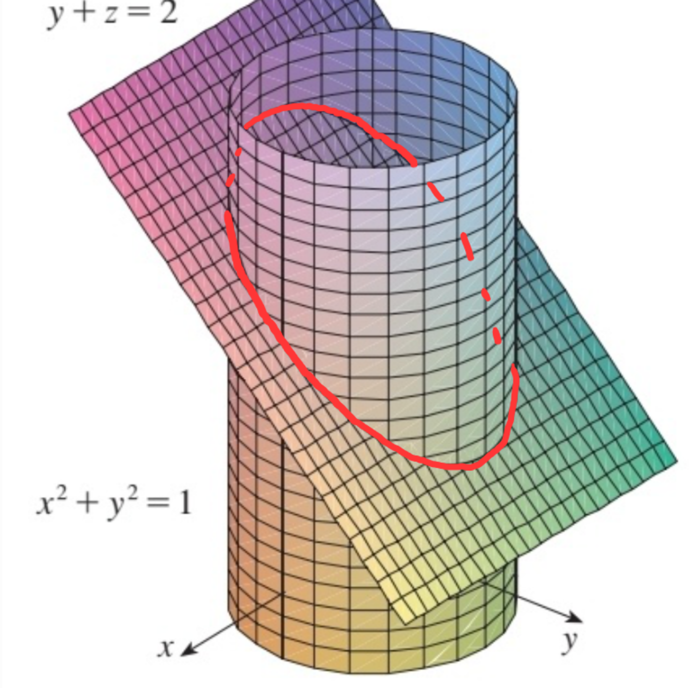
(b) the line in $\mathbb{R}^2$ through $(1,-1)$ and $(2,10)$

(c) the line in $\mathbb{R}^3$ through $(1,0,1)$ and orthogonal to the plane

$$x - 2y + 6z = 7.$$

$\boxed{\text{Ex}}$ Let's parametrize the curve where $x^2+y^2=1$ and $y+z=2$ intersect in $\mathbb{R}^3$; here's a visual:





Since the points all lie on the cylinder, their x- and y-coordinates can be parametrized by

$$x = \cos t \quad \text{and} \quad y = \sin t.$$

On the other hand, z depends on y:

$$z = 2 - y = 2 - \sin t.$$

So the full curve is given by

$$r(t) = \langle \cos t, \sin t, 2 - \sin t \rangle.$$

Notes: (1) The domain of a vector valued function $f = \langle f_1, \dots, f_m \rangle$ is the intersection of the domains of the f_i.

e.g. $f(t) = \langle \sqrt{3-t}, \ln(t) \rangle$

has domain $(0, 3]$.

(2) If $f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$

then for any $P = (a_1, \dots, a_n)$,

$$\lim_{\vec{x} \to P} f(\vec{x}) = \left\langle \lim_{\vec{x} \to P} f_1(\vec{x}), \dots, \lim_{\vec{x} \to P} f_m(\vec{x}) \right\rangle.$$

(3) f is continuous at P exactly when

each f_i is continuous at P.

$\boxed{\text{Def}}$ For a vector valued function

$$f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle,$$

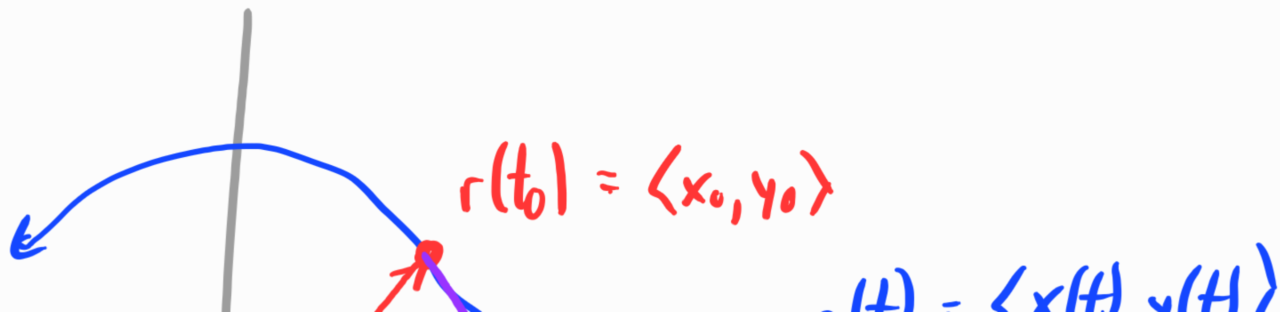
the partial derivative of f with respect to the partial derivative of f with respect to

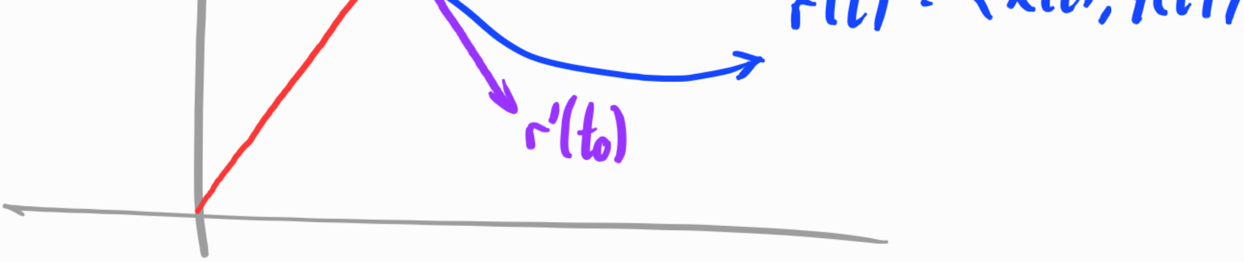
x_i is the vector valued function

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

$$= \left\langle \frac{\partial f_1}{\partial x_i}, \dots, \frac{\partial f_m}{\partial x_1} \right\rangle.$$

$\boxed{\text{Ex}}$ The derivative of a curve $r(t)$ at t_0 is exactly the tangent vector to that curve at the point $r(t_0)$:



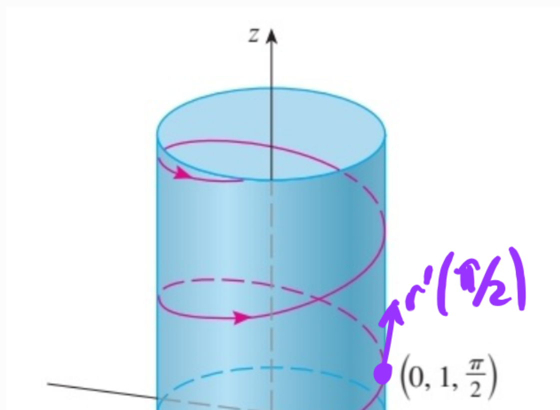


For $r(t) = \langle 1+t^3, te^{-t}, \sin(2t) \rangle$ we have

$$r'(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle.$$

We can also compute $r''(t)$, $r'''(t)$, etc.

Ex The helix $r(t) = \langle \cos t, \sin t, t \rangle$



has derivative $r'(t) = \langle -\sin t, \cos t, 1 \rangle$.

At $t = \frac{\pi}{2}$, $r'\left(\frac{\pi}{2}\right) = \langle -1, 0, 1 \rangle$ and this lets us write down an equation for the tangent line at this point:

$$L(t) = (0, 1, \pi/2) + t\langle -1, 0, 1 \rangle$$

$$= \langle -t, 1, \pi/2 + t \rangle.$$

Exercise 2: Show that at every point on the unit circle in $\mathbb{R}^2$, the tangent line is orthogonal to the ray from the origin to that point.

Exercise 3: Find an equation for the tangent line to each parametric curve at the designated point.

(a) $r(t) = \langle 1+\cos t, 1-\sin t \rangle, \ t=0$

(b) $r(t) = \langle e^t, te^t \rangle, \ t=0$

(c) $r(t) = \langle 2\cos t, \sin t, 2t \rangle, \ t = \frac{\pi}{2}$

(d) $r(t) = \langle t^2+1, \sqrt{t}, e^t \rangle, \ t = 1$

Next time: integrals and vector fields.