# Lecture 13.2

## Last time:

- A vector valued function is a function

$$f = \langle f_1, \dots, f_m \rangle$$

where each $f_j$ depends on some input variables $x_1, \dots, x_n$.

- A curve is a vector valued function with a single input:

$$r(t) = \langle x_1(t), \dots, x_m(t) \rangle.$$

- The partial derivative of $f = \langle f_1, \dots, f_m \rangle$ with respect to one of its input variables

$x_i$ is the vector valued function

$$\frac{\partial f}{\partial x_i} = \left\langle \frac{\partial f_1}{\partial x_i}, \dots, \frac{\partial f_m}{\partial x_i} \right\rangle.$$

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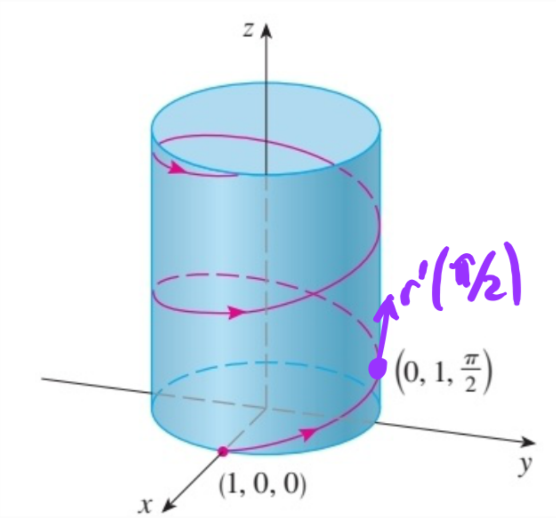
$\boxed{\text{Ex}}$ The helix $r(t) = \langle \cos t, \sin t, t \rangle$




has derivative $r'(t) = \langle -\sin t, \cos t, 1 \rangle$.

At $t = \frac{\pi}{2}$, $r'\left(\frac{\pi}{2}\right) = \langle -1, 0, 1 \rangle$ and this lets us write down an equation for the tangent line at this point:

$$L(t) = (0, 1, \pi/2) + t\langle -1, 0, 1\rangle$$

$$= \langle -t, 1, \pi/2 + t\rangle.$$

Exercise 1: Show that at every point on the unit circle in $\mathbb{R}^2$, the tangent line is orthogonal to the ray from the origin to that point.

Exercise 2: Find an equation for the

tangent line to each parametric curve at the designated point.

(a) $r(t) = \langle 1+\cos t, 1-\sin t \rangle$, $t = 0$

(b) $r(t) = \langle e^t, te^t \rangle$, $t = 0$

(c) $r(t) = \langle 2\cos t, \sin t, 2t \rangle$, $t = \frac{\pi}{2}$

(d) $r(t) = \langle t^2+1, \sqrt{t}, e^t \rangle$, $t = 1$

$\boxed{\text{Ex}}$ For each point along a curve $r(t)$, it may be useful to construct a unit tangent vector (remember: u is a unit vector if $|u| = 1$) at that point.

The formula $T(t) = \dfrac{r'(t)}{|r'(t)|}$ will do this.

For example, if $r(t) = \langle t^2, 2\sin t, 2\cos t \rangle$,

$$r'(t) = \langle 2t, 2\cos t, -2\sin t \rangle$$

$$|r'(t)| = \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{4t^2 + 4} = 2\sqrt{t^2+1}.$$

Then $T(t) = \left\langle \dfrac{t}{\sqrt{t^2+1}}, \dfrac{\cos t}{\sqrt{t^2+1}}, \dfrac{-\sin t}{\sqrt{t^2+1}} \right\rangle.$

Exercise 3: Find a formula for the unit tangent vector to

(a) $r(t) = \langle t^2+1, 3-t^2, t^3 \rangle$

(b) The ellipse $x^2 + 4y^2 = 4$.

Def The **definite integral** of a parametric curve $r(t) = \langle x_1(t), \ldots, x_n(t) \rangle$ from $t=a$ to $t=b$ is

$$\int_a^b r(t)\,dt = \left\langle \int_a^b x_1(t)\,dt, \ldots, \int_a^b x_n(t)\,dt \right\rangle.$$

Ex The integral of

$$r(t) = \langle 2\cos t, \sin t, 2t \rangle$$

from $t=0$ to $t=\pi/2$ is

$$\int_0^{\pi/2} r(t)\,dt = \left\langle \int_0^{\pi/2} 2\cos t\,dt,\ \int_0^{\pi/2} \sin t\,dt,\ \int_0^{\pi/2} 2t\,dt \right\rangle$$

$$= \left\langle 2\sin t \Big|_0^{\pi/2},\ -\cos t \Big|_0^{\pi/2},\ t^2 \Big|_0^{\pi/2} \right\rangle$$

$$= \left\langle 2-0,\ -0+1,\ \left(\frac{\pi}{2}\right)^2 - 0^2 \right\rangle$$

$$= \left\langle 2,\ 1,\ \frac{\pi^2}{4} \right\rangle.$$

**Exercise 4:** Compute $\int_a^b r(t)\,dt$ for each given $r(t)$, $a$ and $b$.

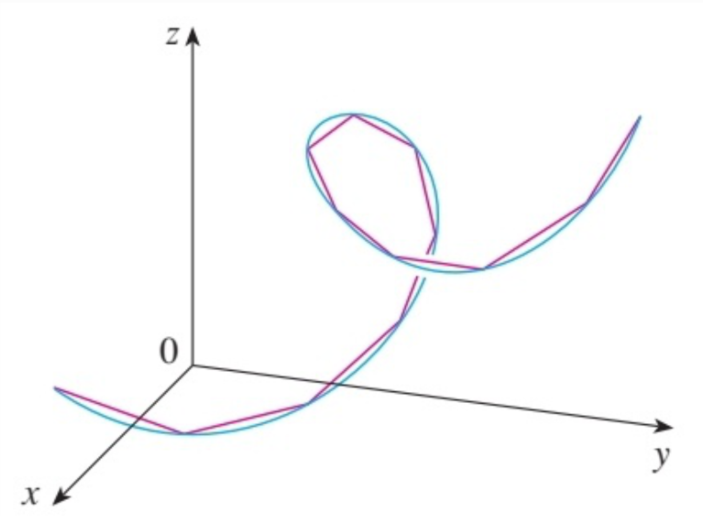
(a) $r(t) = \left\langle \frac{1}{t} + 2t,\ te^{2t},\ -t^2e^{t^3} \right\rangle$, $a=1$, $b=2$

(b) $r(t) = \left\langle \frac{1}{\ldots},\ \frac{t}{\ldots},\ t^2 \ldots \right\rangle$

$\left( \frac{\quad}{1+t}, \frac{\quad}{1+t^2}, \frac{\quad}{1+t^2} \right), \quad a = 0, \ b = 1$

$\boxed{\text{Def}}$ The **arc length** of a curve $r(t)$ from $t = a$ to $t = b$ is

$$L = \int_a^b |r'(t)| \, dt .$$



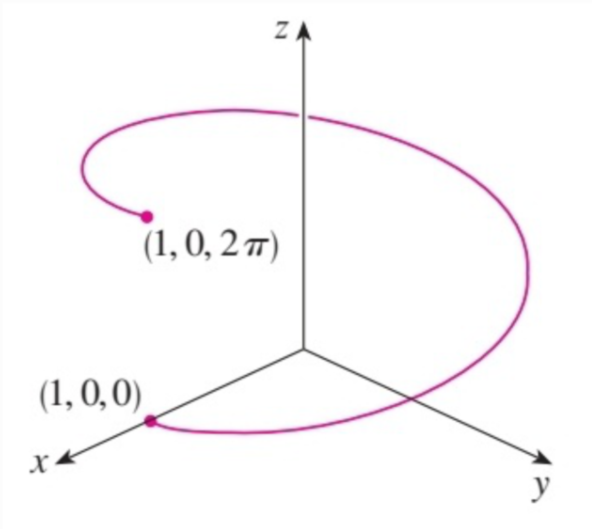

Exercise 5: Describe the arc length of the portion of a circle of radius $r$ from $\theta = a$ to $\theta = b$.

to $\theta = b$.

$\boxed{\text{Ex}}$ Let's find the arc length of the helix

$$r(t) = \langle \cos t, \sin t, t \rangle$$

from $t = 0$ to $t = 2\pi$.




First, $r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}.$$

Then the arc length is

$$L = \int_0^{2\pi} |r'(t)|dt = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= \sqrt{2}\, t \Big|_0^{2\pi} = 2\pi\sqrt{2}.$$

**Exercise 6:** Set up the arc length integral of the curve $r(t) = \langle t, t^2, t^3 \rangle$. Can you solve it?

**Next time:** path integrals.