Lecture 13.2

For the rest of the course, we're going to investigate the following famous result.

Theorem (Dirichlet) For relatively prime integers $a, b \geq 1$, there are infinitely many primes satisfying $p \equiv a \pmod{b}$.

We have already proven a few cases of this, but the full theorem is actually quite deep.

To motivate some of the techniques we will use to prove the Theorem, let's revisit an older theorem:

$\boxed{\text{Theorem}}$ There are infinitely many primes.

This is the $a = b = 1$ case of Dirichlet's Theorem.

Here's a new strategy to prove this fact.

Consider the sum

$$S = \sum_{p \text{ prime}} \frac{1}{p}.$$

If we can show that S diverges, then there must be infinitely many primes.

It's useful to treat S as a special value of a more general expression

$$F(s) = \sum_{p} \frac{1}{p^s}, \quad s \geq 1.$$

Then we'd like to show that $F(s)$

diverges at $s = 1$.

As it happens, $F(s)$ is closely related to one of the most famous functions in number theory:

$\boxed{\text{Def}}$ **Riemann's zeta function** is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Facts:

- $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- $\zeta(s)$ converges for any $s > 1$.

- $\zeta(2)$ converges to $\frac{\pi^2}{6}$ and more generally,

$$\zeta(2k) = \frac{(-1)^{k+1} b_{2k} (2\pi)^{2k}}{2(2k)!}$$

where b_n come from the series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n .$$

- In fact, $\zeta(s)$ can be shown to converge for all <u>complex numbers</u> s with real part $\mathrm{Re}(s) > 1$, but we won't need this for now.

- $\zeta(s)$ satisfies the product formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Exercise 1: Prove the product formula.

Hint: use the Fundamental Theorem of Arithmetic. Don't worry about checking convergence, even though it's important.

Back to our series

$$F(s) = \sum_p \frac{1}{p^s},$$

it is clear that $F(1) < \zeta(1)$.

$$\sum_p \frac{1}{p} < \sum_n \frac{1}{n}$$

However, there's a more subtle relation between $F(s)$ and $\zeta(s)$ revealed by taking the natural logarithm:

$$\log \zeta(s) = \log \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right)$$

$$= \log\left(\prod_p \frac{1}{1-p^{-s}}\right)$$ by Exercise 1

$$= \sum_p -\log\left(1-p^{-s}\right)$$

$$= \sum_p \sum_{n=1}^{\infty} \frac{1}{n} p^{-sn}$$ using the power series expansion for $-\log(1-x)$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \underbrace{\left(\sum_p p^{-sn}\right)}_{F(sn)}$$ since the sum converges absolutely

$$= F(s) + \frac{1}{2}F(2s) + \frac{1}{3}F(3s) + \dots$$

Lemma $G(s) = \sum_{n=2}^{\infty} \frac{1}{n} F(sn)$ converges for all $s \geq 1$.

Pf: For each $n \geq 2$,

$$\frac{1}{n} F(sn) = \frac{1}{n} \sum_{p} \frac{1}{p^{sn}}$$

$$\leq \frac{1}{n} \sum_{m=2}^{\infty} \frac{1}{m^{sn}}$$

$$\leq \frac{1}{n} \int_{1}^{\infty} \frac{1}{x^{2n}} \, dx \qquad \text{by the Integral Test}$$

$$= \frac{1}{n(sn-1)} .$$

Then

$$G(s) = \sum_{n=2}^{\infty} \frac{1}{n} F(sn)$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n(sn-1)}$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges to $\frac{\pi^2}{6}$. $\square$

Putting everything together,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = F(s) + G(s)$$

and $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges while

$G(1)$ converges by the Lemma.

Therefore $F(1) = \sum_p \frac{1}{p}$ diverges.

How can we adapt this strategy to prove Dirichlet's Theorem?

Consider the series

$$F(s) = F_{a,b}(s) = \sum_{p \equiv a \pmod b} \frac{1}{p^s}.$$

We can prove Dirichlet's Theorem by

showing that $F(s)$ di[illegible]

showing that $F(s)$ diverges at $s = 1$.

What's the right analogue of the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$?

We want to see $\sum_{p \equiv a \pmod b} \frac{1}{p^s}$ among the terms of some infinite series, so we could try

$$\sum_{n \equiv a \pmod b} \frac{1}{n^s} .$$

In fact, these terms all appear in

$\zeta(s)$, so it makes sense to put them all on the same footing.

$\boxed{\text{Def}}$ A **Dirichlet series** is a function of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}$$

where $f: \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetic function.

$\boxed{\text{Ex}}$ 1 $\zeta(s)$ is the Dirichlet series for the arithmetic function $f(n) = 1$.

② Each of the factors

$$\frac{1}{1-p^{-s}} = \sum_{k=0}^{\infty} p^{-ks}$$

is a Dirichlet series, namely for

$$f_p(n) = \begin{cases} 1, & n = p^k \text{ for } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

That is, f_p is the "p-power indicator function.

3. For any subset $S \subseteq \mathbb{N}$, let

$$f_S(n) = \begin{cases} 1, & n \in S \\ 0, & n \notin S. \end{cases}$$

Then

$$\sum_{n \in S} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{f_S(n)}{n^s}.$$

For our purposes, we can take

$$S = \{ n \in \mathbb{N} \mid n \equiv a \pmod{b} \}$$

to get an arithmetic function

$$f(n) = f_{a,b}(n) = \begin{cases} 1, & n \equiv a \pmod{b} \\ 0, & n \not\equiv a \pmod{b}. \end{cases}$$

To relate $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ to our sum

$$F(s) = \sum_{p \equiv a \pmod{b}} \frac{1}{p^s}$$

we need:

- a product formula

- further properties of arithmetic functions.

Next time: characters and L-functions.