# Lecture 14.2

## Last time:

- To find the domain of $f(x_1, \dots, x_n)$, check the usual things: square roots, denominators, ln.

- A cross section of $f(x,y)$ is a curve resulting from slicing $z = f(x,y)$ with a plane.

- A level curve of $f(x,y)$ is a horizontal cross section, given by

$$f(x,y) = k$$

for some number $k$.

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## Limits

**Q:** How does the concept of a limit

$$\lim_{x \to a} f(x)$$

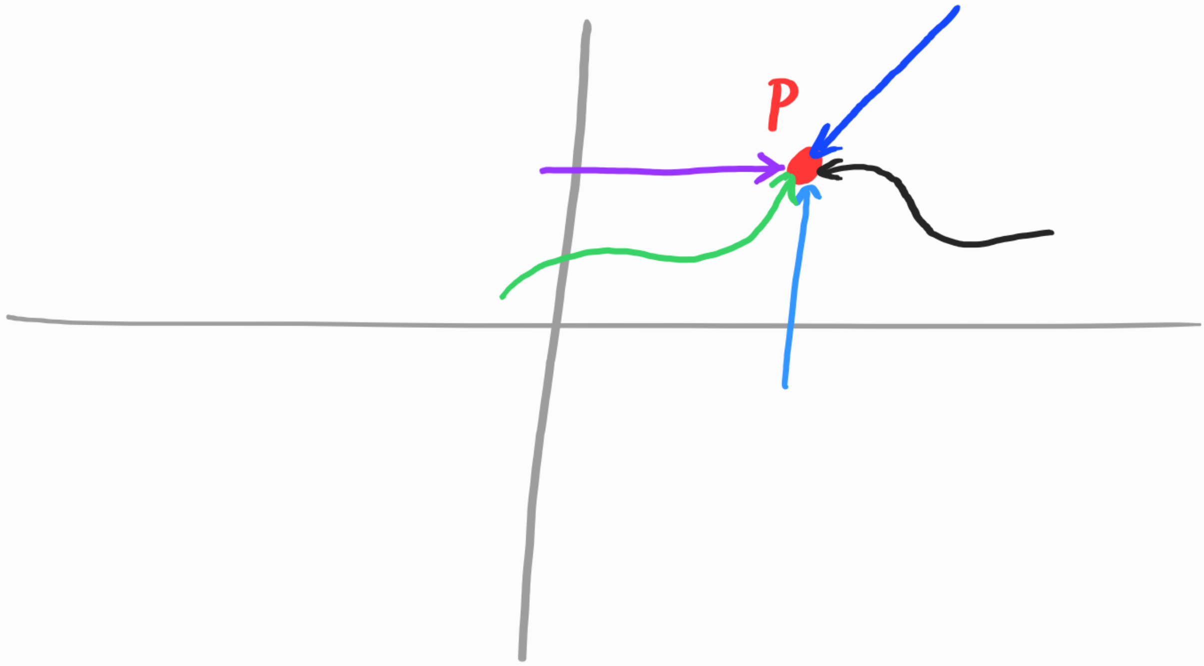
translate to multiple variables?

Loosely, we want "the values of $f$ to approach $L$ as the inputs approach a single point".

Problem: in 2+ dimensions, there are

of approaching a point

many ways of approaching a point.

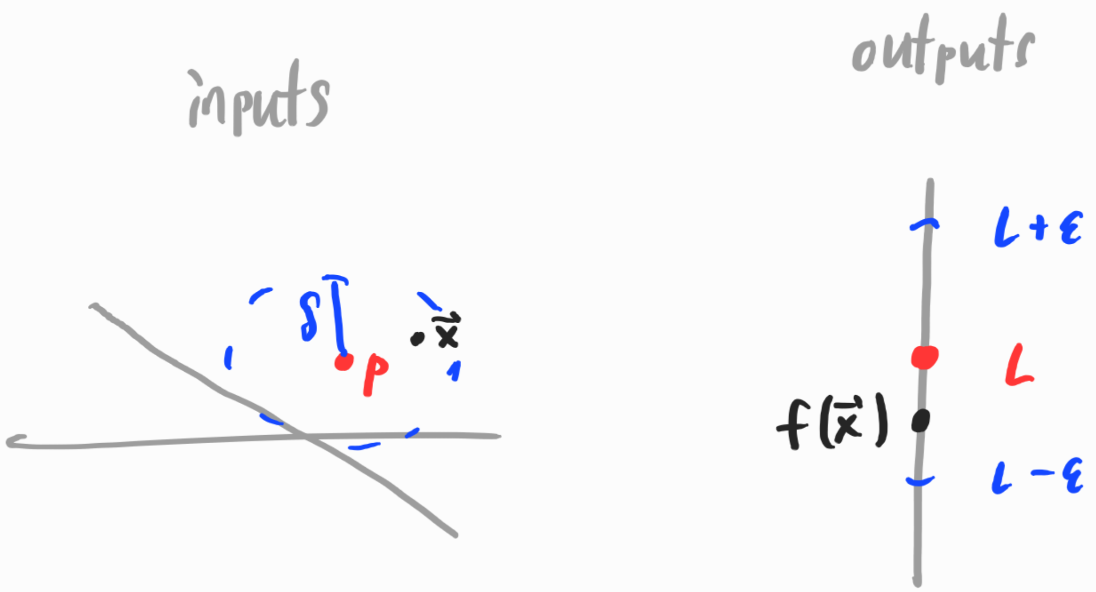



$\boxed{\text{Def}}$ For a function $f(x_1, \dots, x_n)$, $f$ has a **limit** $L$ at $P = (p_1, \dots, p_n)$ if, no matter how close to $L$ we want to get, there is a sufficiently small sphere around $P$ such that $f(\vec{x})$ is close to $L$ for all $\vec{x}$ inside the sphere. In symbols,

$$\lim_{(x_1,\ldots,x_n)\to(p_1,\ldots,p_n)} f(x_1,\ldots,x_n) = L$$

if for every small $\varepsilon > 0$, there is a radius $\delta > 0$ such that

if $|\vec{x}P| < \delta$ then $|f(\vec{x}) - L| < \varepsilon$.




In $\mathbb{R}^2$, this translates to:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$, there's a $\delta > 0$

such that if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$

then $|f(x,y) - L| < \varepsilon$.

How can we evaluate limits like this?

$\boxed{\text{Ex}}$ Here's a table of outputs for

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

near $(x,y) = (0,0)$:

| $x \backslash y$ | −1.0 | −0.5 | −0.2 | 0 | 0.2 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|---|
| −1.0 | 0.455 | 0.759 | 0.829 | 0.841 | 0.829 | 0.759 | 0.455 |
| −0.5 | 0.759 | 0.959 | 0.986 | 0.990 | 0.986 | 0.959 | 0.759 |
| −0.2 | 0.829 | 0.986 | 0.999 | 1.000 | 0.999 | 0.986 | 0.829 |
| 0 | 0.841 | 0.990 | 1.000 | | 1.000 | 0.990 | 0.841 |
| 0.2 | 0.829 | 0.986 | 0.999 | 1.000 | 0.999 | 0.986 | 0.829 |
| 0.5 | 0.759 | 0.959 | 0.986 | 0.990 | 0.986 | 0.959 | 0.759 |
| 1.0 | 0.455 | 0.759 | 0.829 | 0.841 | 0.829 | 0.759 | 0.455 |

It appears $\lim_{(x,y)\to(0,0)} f(x,y) = 1.$

We will learn a technique to verify this limit soon.

$\boxed{\text{Ex}}$ Here's a table of outputs for

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

near $(0,0)$ :

| $x \backslash y$ | −1.0 | −0.5 | −0.2 | 0 | 0.2 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|---|
| −1.0 | 0.000 | 0.600 | 0.923 | 1.000 | 0.923 | 0.600 | 0.000 |
| −0.5 | −0.600 | 0.000 | 0.724 | 1.000 | 0.724 | 0.000 | −0.600 |
| −0.2 | −0.923 | −0.724 | 0.000 | 1.000 | 0.000 | −0.724 | −0.923 |
| 0 | −1.000 | −1.000 | −1.000 | | −1.000 | −1.000 | −1.000 |
| 0.2 | −0.923 | −0.724 | 0.000 | 1.000 | 0.000 | −0.724 | −0.923 |
| 0.5 | −0.600 | 0.000 | 0.724 | 1.000 | 0.724 | 0.000 | −0.600 |
| 1.0 | 0.000 | 0.600 | 0.923 | 1.000 | 0.923 | 0.600 | 0.000 |

Here, it appears $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Let's show this carefully.

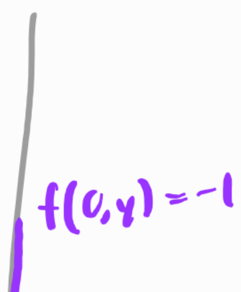
Notice that along the line $x = 0$, the $z$-values appear stable. We can write this as a limit along the path $x = 0$:

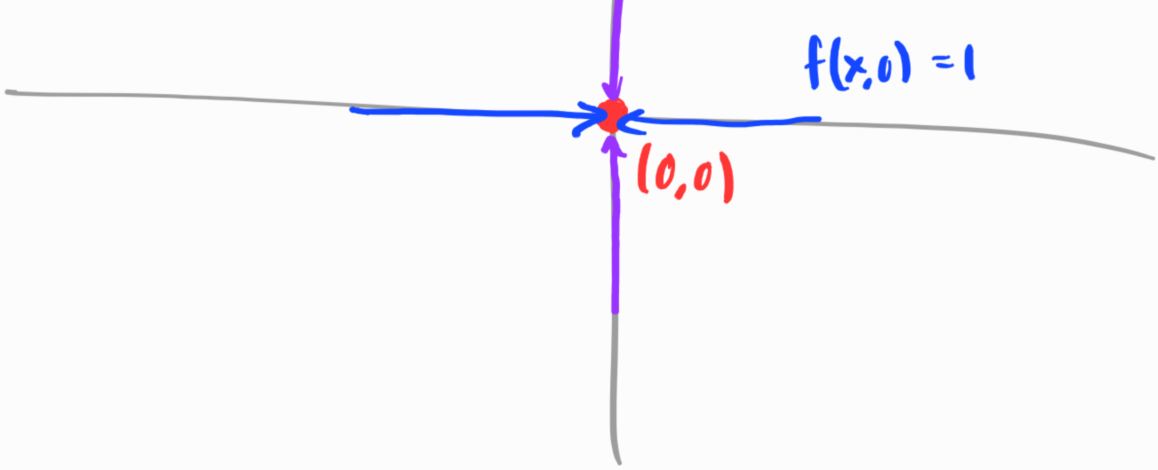
$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1.$$

On the other hand, approaching $(0,0)$ along the path $y=0$ yields different behavior:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{y\to 0} 1 = 1.$$

This shows that with any circle centered at $(0,0)$, there are points $(x,y)$ with $f(x,y) = -1$ and also points with $f(x,y) = 1$, so the outputs cannot converge to a single value.





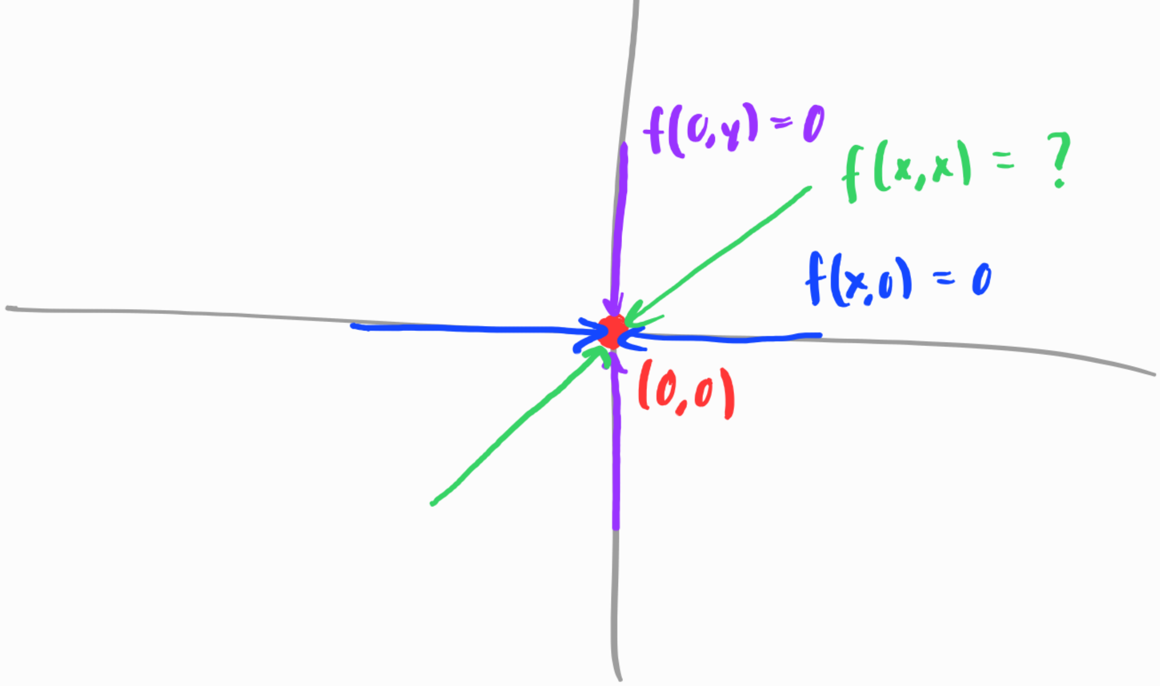

Ex Let's study the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

in a similar way.

You can show that along the lines $y=0$ and $x=0$, the outputs are all $0$.

However, in $\mathbb{R}^2$ there are many other paths of approach, such as




Along another straight path, such as $y = x$, the function has different behavior:

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

This shows that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE.

$\boxed{\text{Ex}}$ Sometimes even lines don't tell the full story. Consider

$$\lim \quad \frac{xy^2}{\ldots}$$

$(x,y)\to(0,0) \quad x^2+y^4 \,.$

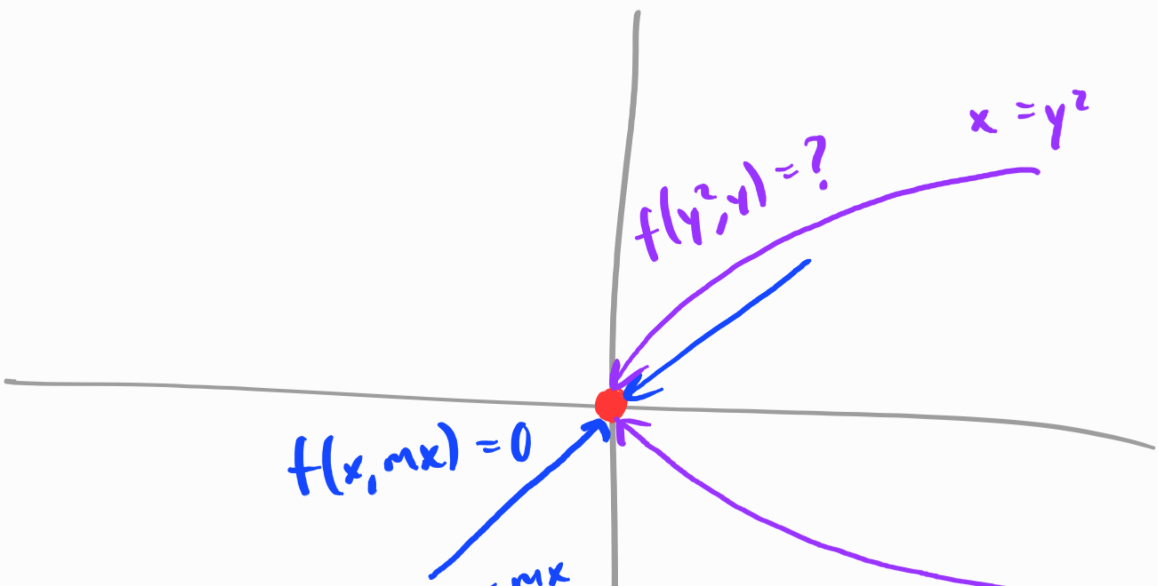
Along any line $y = mx$ passing through $(0,0)$, we get:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0} \frac{m^2 x}{1+m^4x^2} = 0.$$

However, approaching along a parabola such as $x = y^2$ produces a different result:



$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{(y^2)y^2}{(y^2)^2+y^4}$$

$$= \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

So the limit does not exist.

How do we show that a limit does exist in general?

$\boxed{\text{Def}}$ A function $f(x_1,\dots,x_n)$ is continuous at $P=(p_1,\dots,p_n)$ if:

(1) $f(p_1,\dots,p_n)$ exists ($P$ is in the domain),

(2) $\lim\limits_{(x_1,\dots,x_n)\to\dots} f(x_1,\dots,x_n)$ exists, and

$(x_1, \ldots, x_n) \to p$

(3) $\lim_{(x_1,\ldots,x_n)\to p} f(x_1, \ldots, x_n) = f(p_1, \ldots, p_n)$.

$\boxed{\text{Prop}}$ The following functions are continuous on their domains:

(a) Sums, differences and products of continuous functions.

(b) Polynomials $p(x_1, \ldots, x_n)$.

(c) Rational functions $\dfrac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)}$ polynomials

on their domain, i.e. wherever $q \neq 0$.

(d) nth roots $\sqrt[n]{f(x_1, \ldots, x_n)}$ wherever $f$ is continuous and, if $n$ is even where $f \geq 0$.

where $r \geq 0$.

(e) Exponential functions.

(f) Trig functions and logarithms on their domain.

$\boxed{\text{Theorem (Squeeze Theorem)}}$ Suppose that

$$f(x_1,\dots,x_n) \leq g(x_1,\dots,x_n) \leq h(x_1,\dots,x_n)$$

for all $(x_1,\dots,x_n)$ in some sphere around $P = (p_1,\dots,p_n)$ and

$$\lim_{(x_1,\dots,x_n)\to(p_1,\dots,p_n)} f(x_1,\dots,x_n)$$

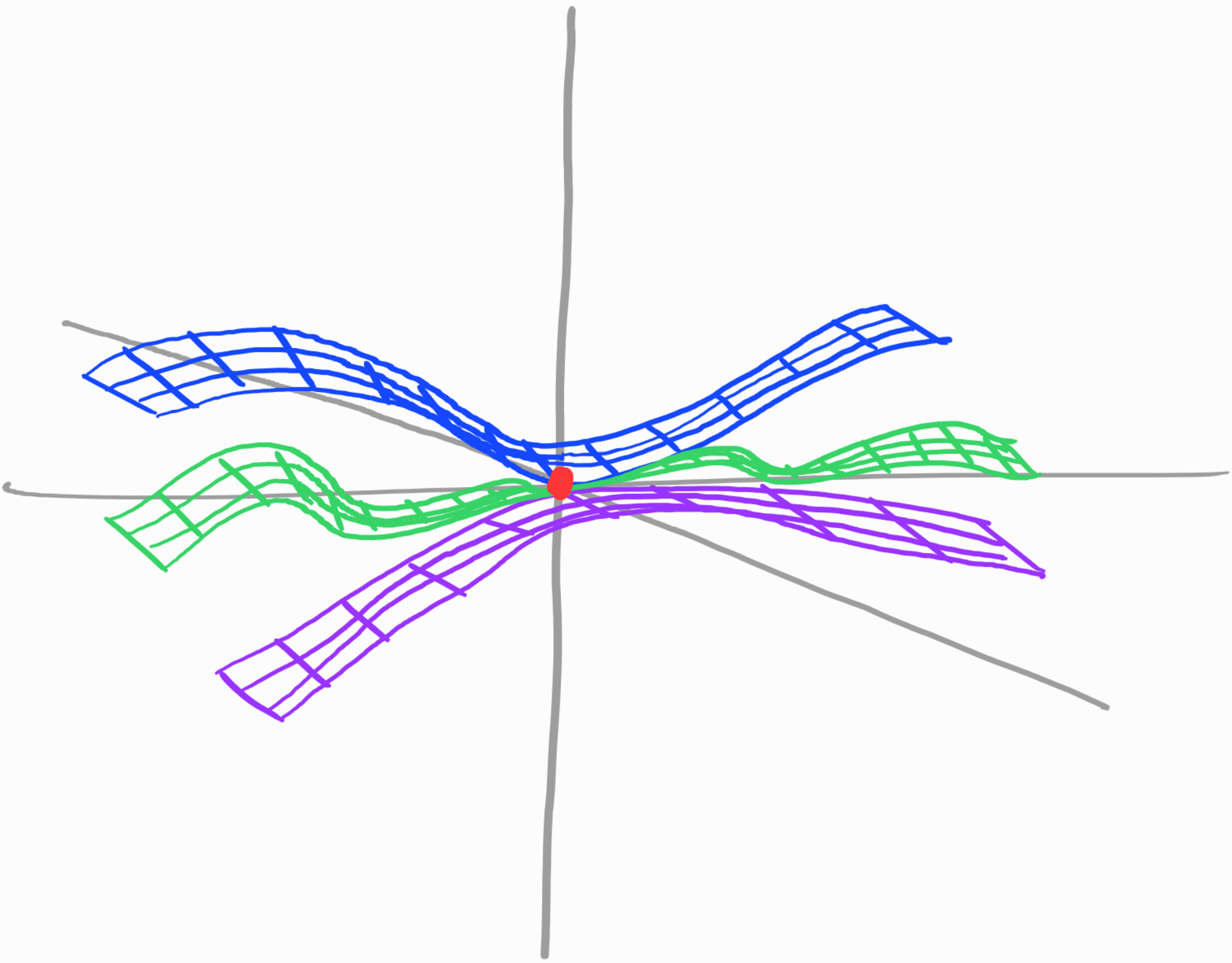
and $\lim_{(x_1,\dots,x_n)\to(p_1,\dots,p_n)} h(x_1,\dots,x_n)$

$(x_1,\dots,x_n)\to(p_1,\dots,p_n)$

both exist and equal the same value $L$.

Then $\displaystyle\lim_{(x_1,\dots,x_n)\to(p_1,\dots,p_n)} g(x_1,\dots,x_n)$ also

exists and equals $L$.



**Exercise 1:** Compute each limit or state that it doesn't exist.

**(a)** $$\lim_{(x,y)\to(1,1)} \frac{2x^2-xy-y^2}{x^2-y^2}$$

Hint: try approaching along the line $y=1$ first. Can you adapt your technique to the general limit?

**(b)** $$\lim_{(x,y)\to(0,0)} x^2\sin\left(\frac{1}{x^2+y^2}\right)$$

**(c)** $$\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}}$$

**Exercise 2:** Find $A$ so that

$$f(x,y) = \begin{cases} \dfrac{x^2 - 2xy}{x^2 - 4y^2}, & x \neq \pm 2y \\ A, & (x,y) = (2,1) \end{cases}$$

is continuous at $(2,1)$.

**Exercise 3:** Find where

$$f(x,y) = \begin{cases} \dfrac{\cos(y)\sin(x)}{x}, & x \neq 0 \\ \cos(y), & x = 0 \end{cases}$$

is continuous.

Next time: partial derivatives.