Lecture 14.3

Last time:

- $\lim_{\vec{x} \to P} f(\vec{x}) = L$ if for any window $(L-\varepsilon, L+\varepsilon)$ around the target L, there is a sphere of radius δ centered at P such that for all $\vec{x}$ inside the sphere, i.e. $|\vec{x}P| < \delta$, $f(\vec{x})$ lies in the target window, i.e. $|f(\vec{x}) - L| < \varepsilon$.

- In $\mathbb{R}^n$ for $n \geq 2$, there are many different paths to approach P and certain paths may produce different limits.

- The only surefire ways to verify a limit are:
  - do some algebra
  - recognize the function as one of our examples of continuous functions
  - use the Squeeze Theorem.

Partial Derivatives

In $\mathbb{R}^n$ for $n \geq 2$, there are many ways of moving around within the domain of a function.

Partial derivatives measure the rate of change of a function $f(x_1, \dots, x_n)$ as each input x_i changes independently.

$\boxed{\text{Def}}$ For a function $f(x_1, \dots, x_n)$ and a point $P = (a_1, \dots, a_n)$, the **partial derivative** of f with respect to x_i $(1 \leq i \leq n)$ at P is the instantaneous rate of change of $f(a_1, \dots, x_i, \dots, a_n)$ at $x_i = a_i$:

$$\frac{\partial f}{\ } (\quad \lim \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{}$$

$$\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = \lim_{h \to 0} \frac{\qquad}{h}$$

$$= \frac{\text{"change in } f\text{"}}{\text{"change in } x_i\text{"}}$$

For functions $f(x,y)$, we will write

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

NOTE: If we let a and b vary, we

get two functions, $f_x(x,y)$ and $f_y(x,y)$.

$\boxed{\text{Ex}}$ For $f(x,y) = y^2 - x^2 + 3x\sin y$,

we have:

$$f_x = -2x + 3\sin y$$

$$f_y = 2y + 3x\cos y.$$

Interpretation: $f_x(a,b)$ is the slope of the tangent line to the cross section

$z = f(x, b)$: (that is, hold $y = b$ constant)

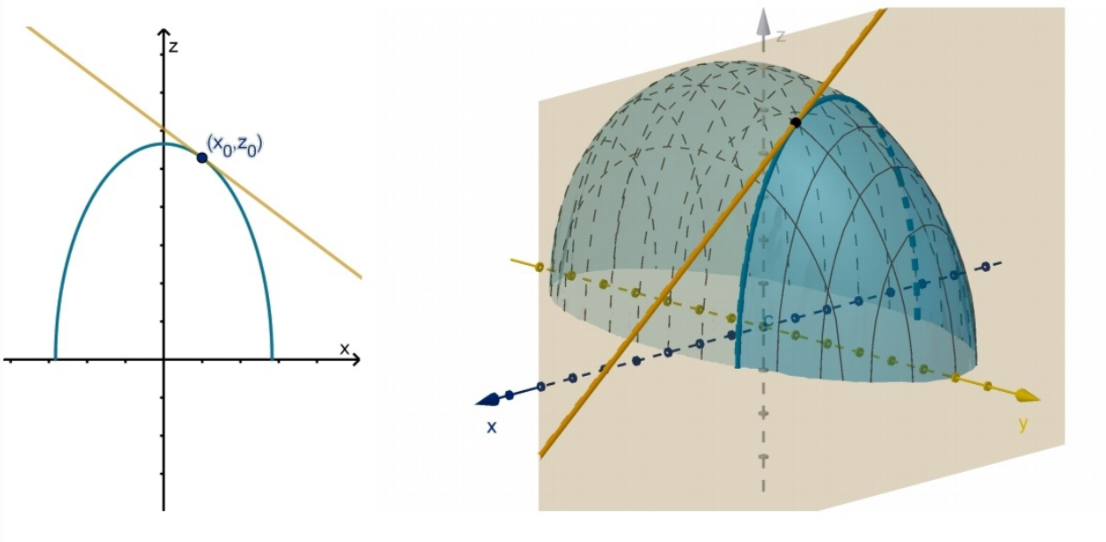
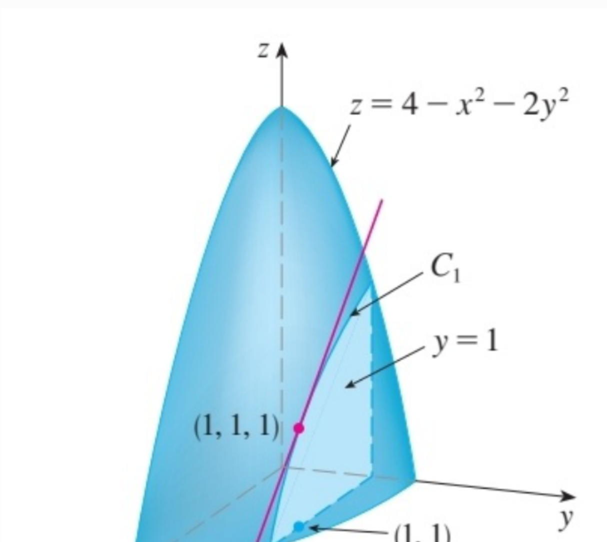
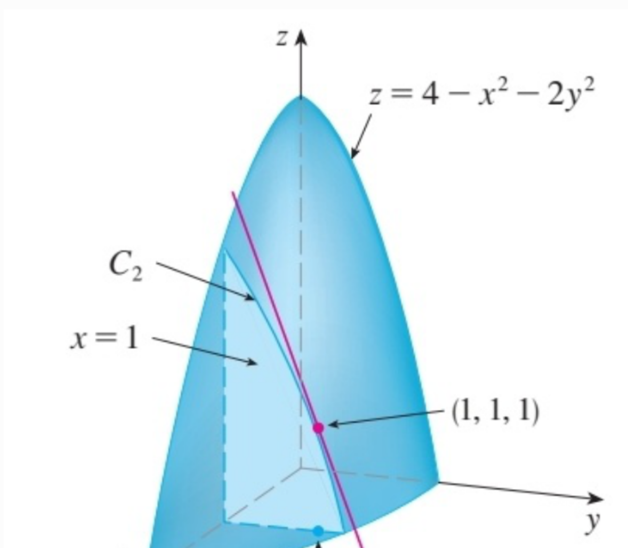
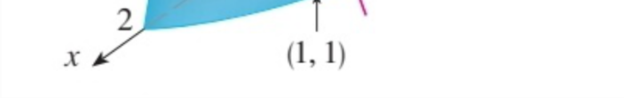


Figure: The tangent line to $z = f(x, y)$ in the x direction

$\boxed{\text{Ex}}$ For $f(x,y) = 4 - x^2 - 2y^2$, the values $f_x(1,1)$ and $f_y(1,1)$ look like:







Let's find them ourselves. First,

$$f_x(x,y) = -2x$$

$$f_x(1,1) = -2.$$

Next,

$$f_y(x,y) = -4y$$

$$f_y(1,1) = -4.$$

Are these reflected in the images?

$\boxed{\text{Ex}}$ For $f(x,y) = \sqrt{xy}$,

$$f_x = \frac{1}{2}(xy)^{-1/2} \cdot y$$ Chain Rule

$$= \frac{y}{2\sqrt{xy}} = \frac{\sqrt{y}}{2\sqrt{x}}.$$

$\boxed{\text{Ex}}$ For $f = x^2 - xy + \cos(yz) - 5z^3$,

$$f_x = 2x - y$$

$$f_y = -x - z\sin(yz)$$

$$f_z = -y\sin(yz) - 15z^2.$$

Exercise 1 : Compute the partial

derivatives of

(a) $f(x,y) = x^2 + 4xy + y^3 + 4y$

(b) $g(x,y) = \ln(e^{xy} + x^2 + 2y^4 + 1)$

Exercise 2: If $f = x^2 + 2y^2 - 2x$,

find where $f_x = 0$ and $f_y = 0$.

Interpret these points graphically.

Just like a single variable function

f(x) has a 2nd, 3rd, etc. derivative,

it is possible to take higher order partial derivatives of multivariable functions:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

notice the order changes

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

Higher order derivatives are possible, e.g.

f_{xxx}, f_{xyxy}, $f_{xxxxxyyyy}$, etc.

$\boxed{\text{Ex}}$ For $f = \sin(3x + x^2y)$, we have

$$f_x = (3 + 2xy)\cos(3x + x^2y)$$

$$f_{xx} = 2y\cos(3x + x^2y) - (3 + 2xy)^2\sin(3x + x^2y)$$

$$f_{xy} = 2x\cos(3x + x^2y) - (3 + 2xy)x^2\sin(3x + x^2y).$$

Exercise 3 : Compute f_{yy} and f_{yx} for the same example. What do you notice?

$\boxed{\text{Theorem}}$ If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

More generally, if all partial derivatives are continuous, then the order in which we differentiate the variables does not matter.

Exercise 4: Compute f_{xx}, f_{yy} and f_{xy} for $f = (x+2)(y-1)e^{x^2+y^2}$.

Exercise 5: What is f_{yxyx} for

$$f = 11 - y + y^4x^2 + 12x^{2022} + y^{2023} \, ?$$

Next time: tangent planes.