Lecture 14.4

Last time:

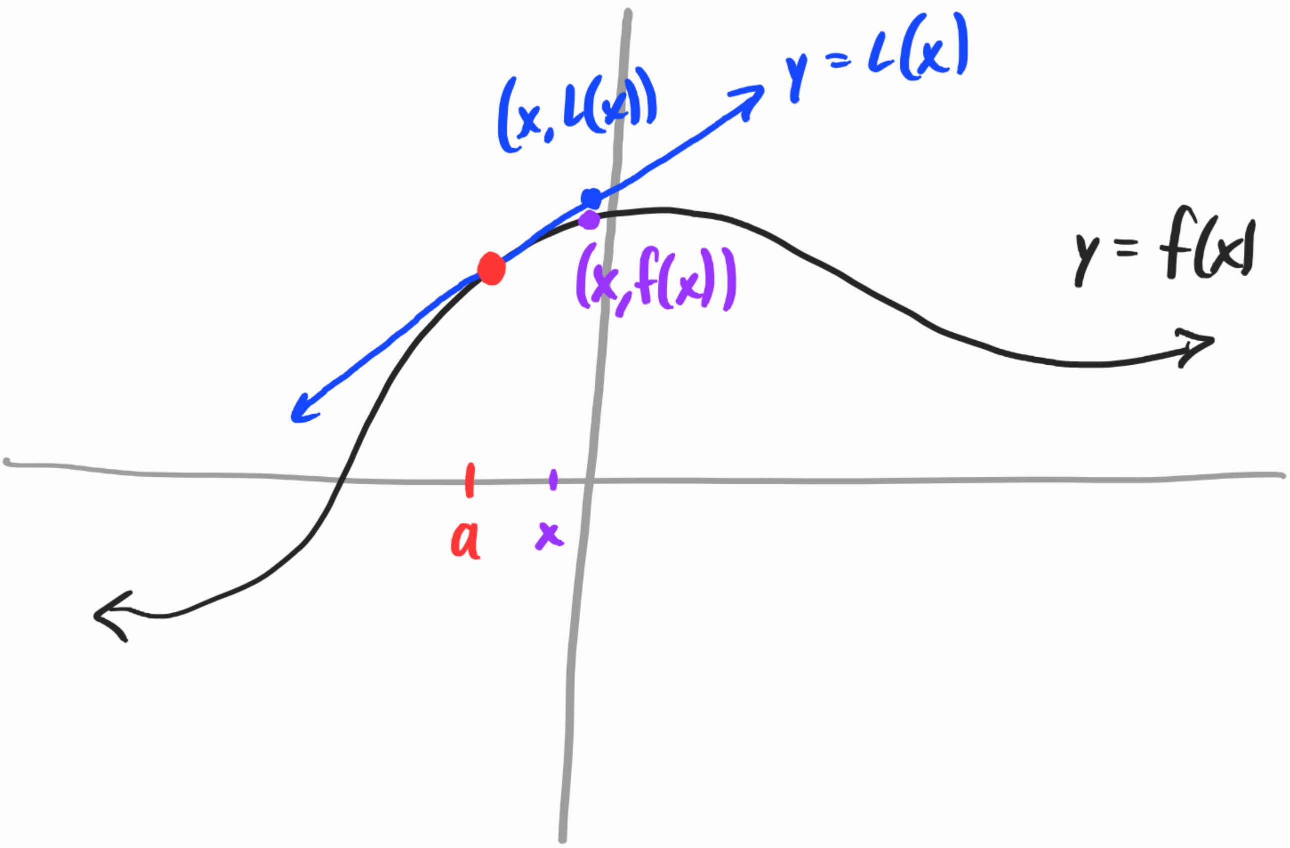
- The partial derivative of $f(x_1,\ldots,x_n)$ with respect to x_i is

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\ldots,x_i+h,x_n) - f(x_1,\ldots,x_n)}{h}.$$

- To compute f_{x_i}, hold all variables constant except x_i and take the single variable derivative with respect to x_i.

Tangent Planes

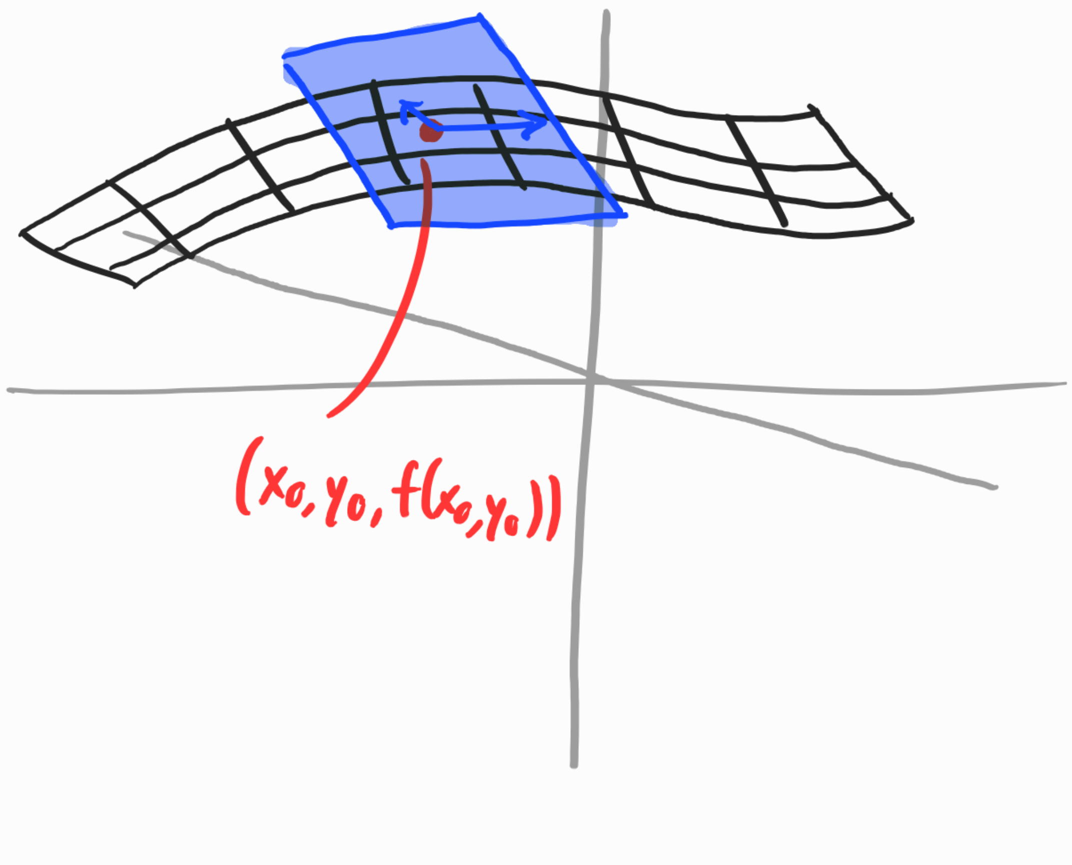
In single variable calculus, a function's derivative helps define the tangent line at any point, which is a good approximation to the function.



$$L(x) \approx f(x)$$

For $f(x, y)$ we have two partial

For $f(x,y)$, we have two partial derivatives, f_x and f_y, which together define a tangent <u>plane</u> that closely approximates the values of $f(x,y)$.



To find an equation for the tangent plane, let's suppose it has a normal equation

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

Dividing by c, we can rewrite this as

$$z - z_0 = A(x-x_0) + B(y-y_0)$$

where $A = \frac{-a}{c}$, $B = \frac{-b}{c}$.

In the plane $y = y_0$, this has cross section

$$z - z_0 = A(x-x_0)$$

which is also the tangent line to

$z = f(x, y_0)$, so $A = f_x(x_0, y_0)$,

the partial derivative of f with

respect to x at (x_0, y_0).

Similarly, $B = f_y(x_0, y_0)$.

Tangent Plane Equation:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Alternatively, the tangent lines at (x_0, y_0)

have vector equations

$(x, y) = (x_0, y_0) + t\vec{v}$ and $(x, y) = (x_0, y_0) + t\vec{w}$

where $\vec{v} = \langle 1, 0, f_x(x_0, y_0) \rangle$

and $\vec{w} = \langle 0, 1, f_y(x_0, y_0) \rangle$.

Exercise 1: Compute $\vec{v} \times \vec{w}$ and show that it gives the same normal equation for the tangent plane.

$\boxed{\text{Ex}}$ Let's find the tangent plane to

$$z = 2x^2 + y^2$$

at $(1, 1, 3)$. The partial derivatives

we need are:

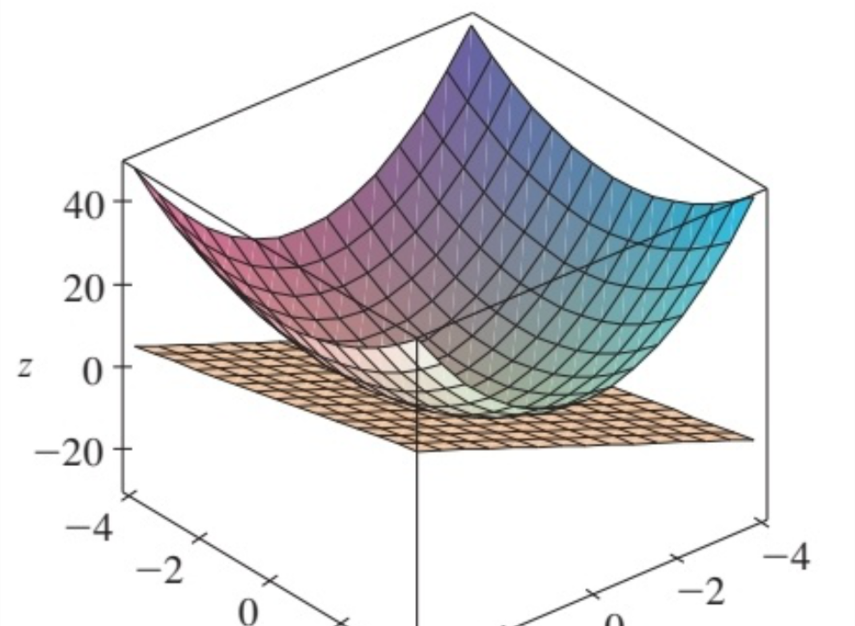
$$\frac{\partial z}{\partial x} = 4x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

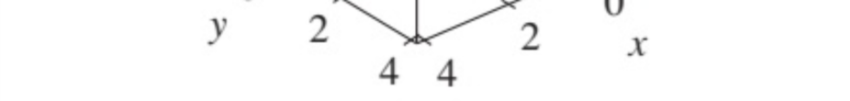
which at $(x,y) = (1,1)$ have the values

$$\frac{\partial z}{\partial x}(1,1) = 4 \quad \text{and} \quad \frac{\partial z}{\partial y}(1,1) = 2.$$

The tangent plane then has the equation

$$z - 3 = 4(x-1) + 2(y-1).$$





The linearization of a function $f(x,y)$ is the linear function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

formed by solving for z in the tangent plane equation.

For (x,y) near (x_0, y_0), $L(x,y)$ will be reasonably close to $f(x,y)$.

$\boxed{\text{Ex}}$ Let's linearize $f(x,y) = xe^{xy}$

at $(1,0)$ and use it to

approximate $f(1.1,-0.1)$. First,

$$f_x = e^{xy} + xye^{xy} \text{ and } f_y = x^2 e^{xy}.$$

Then $f_x(1,0) = 1$ and $f_y(1,0) = 1$ so

$$L(x,y) = 1 + (x-1) + y.$$

At $(1.1,-0.1)$,

$$L(1.1,-0.1) = 1 + (1.1-1) - 0.1 = 1.$$

Compare this to the true value of $f(1.1,-0.1)$:

$$1.1e^{-0.11} \approx 0.9854.$$

Q: How far off from the true function value will a linear approximation be?

Def The total differential of a function $f(x,y)$ is the expression

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Here, the symbols dx, dy and dz represent "small changes in x, y and z".

More precisely, for any point (x,y)

near (x_0, y_0), $dx = x - x_0$, $dy = y - y_0$

and

$$dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

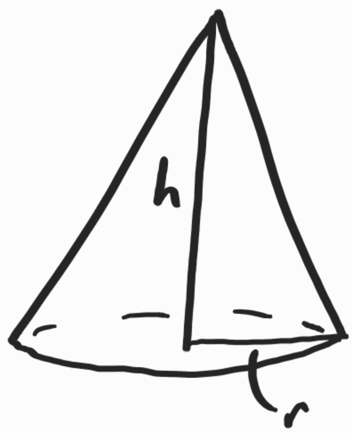
$$= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= z - z_0.$$

So dz measures the approximate difference between $z = f(x,y)$ and $z_0 = f(x_0, y_0)$.

$\boxed{\text{Ex}}$ The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the circular base and

h is the height.



If $r \approx 10\text{cm}$ and $h \approx 25\text{cm}$, with

an error tolerance of up to 0.1cm

each, let's estimate the maximum

error in the reported volume.

By definition,

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

where $$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \quad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2.$$

In this case, at $(r, h) = (10, 25)$,

$$\frac{\partial V}{\partial r} = \frac{500\pi}{3} \quad \text{and} \quad \frac{\partial V}{\partial h} = \frac{100\pi}{3}$$

so $$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1)$$

$$= 20\pi \approx 63 \text{ cm}^3.$$

The volume is estimated to be

$$V = \frac{2500\pi}{3} \approx 2618 \text{ cm}^3$$

so with the error estimate, we can say with confidence the true volume is between $2555\,cm^3$ and $2681\,cm^3$.

Exercise 2: Linearize $f(x,y) = \sqrt{x}e^y$ and approximate $\sqrt{4.02e^{0.05}}$. What is the maximum error in your approximation?

Exercise 3: Find an equation for the tangent plane to $z = \ln(2x+y)$

at $(-1, 3)$.

Exercise 4: Find the linearization of $f(x,y) = 4x^2 - ye^{2x+y}$ at $(-2, 4)$.

Next time: the Chain Rule.