Lecture 14.5

Last time:

- The tangent plane to $z = f(x,y)$ at (x_0, y_0) has equation

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

- Solving for z, we get a function

$$L(x,y) = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

which closely approximates values of $f(x,y)$ near (x_0, y_0).

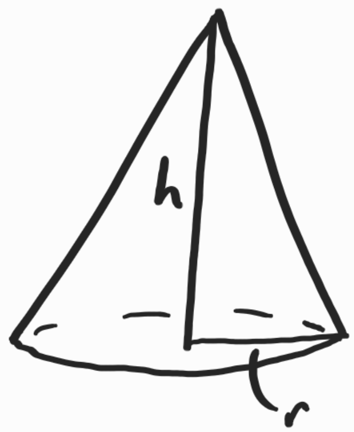
- The total differential of $f(x,y)$ is

The total differential of $f(x,y)$ is

$$dz = df = f_x\,dx + f_y\,dy.$$

In plain terms, dz measures the approximate difference between $z = f(x,y)$ and $z_0 = f(x_0, y_0)$. Here's an example to illustrate how dz controls the error in an approximation.

Ex The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the circular base and h is the height.



If $r \approx 10$ cm and $h \approx 25$ cm, with an error tolerance of up to 0.1cm each, let's estimate the maximum error in the reported volume.

By definition,

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

where $\dfrac{\partial V}{\partial r} = \dfrac{2}{3}\pi r h$, $\dfrac{\partial V}{\partial h} = \dfrac{1}{3}\pi r^2$.

In this case, at $(r, h) = (10, 25)$,

$$\frac{\partial V}{\partial r} = \frac{500\pi}{3} \quad \text{and} \quad \frac{\partial V}{\partial h} = \frac{100\pi}{3}$$

so
$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1)$$
$$= 20\pi \approx 63 \text{ cm}^3.$$

The volume is estimated to be

$$V = \frac{2500\pi}{3} \approx 2618 \text{ cm}^3$$

so with the error estimate, we

can say with confidence the true volume is between $2555\,cm^3$ and $2681\,cm^3$.

$\boxed{\text{The Chain Rule}}$

Recall that for a single variable function $y = f(u)$ where $u = u(x)$ is itself a function of x, we have

$$\frac{dy}{dx} = f'(u)\,u'(x)$$

or $$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

In higher dimensions, a similar rule exists:

Chain Rule If $z = f(x,y)$ is a differentiable function and $x = x(t)$ and $y = y(t)$ are both single variable functions of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Note: we use d for derivatives of

single variable expressions and ∂ for partial derivatives.

$\boxed{\text{Ex}}$ Let's compute $\frac{df}{dt}$ where

$$f(x,y) = xe^{xy}, \quad x = t^2, \quad y = \frac{1}{t}.$$

By the Chain Rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

The derivatives we need are:

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}$$

∂f

$$\frac{\partial f}{\partial y} = x^2 e^{xy}$$

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = \frac{-1}{t^2}.$$

Then

$$\frac{df}{dt} = \left(e^{xy} + xye^{xy}\right)2t - x^2 e^{xy} \cdot \frac{1}{t^2}$$

$$= \left(e^t + te^t\right)2t - t^2 e^t$$

$$= (2t + t^2)e^t.$$

You can confirm this by substituting $x = t^2$ and $y = \frac{1}{t}$ into the original expression for f and computing

$f'(t)$ the old fashioned way, but it's often simpler to use the multivariable chain rule.

There's no reason x and y can't be multivariable functions themselves, say $x(s,t)$ and $y(s,t)$.

Chain Rule II If $z = f(x,y)$ is a differentiable function and $x = x(s,t)$

and $y = y(s,t)$ are also differentiable,

then
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

and
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

$\boxed{\text{Ex}}$ For $f(x,y) = e^{2x}\sin(3y)$ with $x = st - t^2$ and $y = \sqrt{s^2+t^2}$, we have the following partial derivatives:

$$f_x = 2e^{2x}\sin(3y)$$

$$f_y = 3e^{2x}\cos(3y)$$

$$x_s = t, \quad x_t = s - 2t$$

$$y_s = \frac{\;}{\sqrt{s^2+t^2}}, \quad y_t = \frac{t}{\sqrt{s^2+t^2}}.$$

Then

$$f_s = f_x x_s + f_y y_t$$

$$= 2e^{2x}\sin(3y)t + 3e^{2x}\cos(3y)\frac{s}{\sqrt{s^2+t^2}}$$

$$= 2te^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)$$

$$+ \frac{3se^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}.$$

We can also do implicit differentiation in multiple variables.

Given an implicit equation

$$F(x,y) = 0$$

this is a level curve!

we can treat $y = y(x)$ and use the chain rule to write

$$0 = \frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}.}$$

$\boxed{\text{Ex}}$ Find $\dfrac{dy}{dx}$ at $(3,3)$ where

$$x^3 + y^2 - 4xy = 0.$$

Here, $F(x,y) = x^3 + y^2 - 4xy$ has

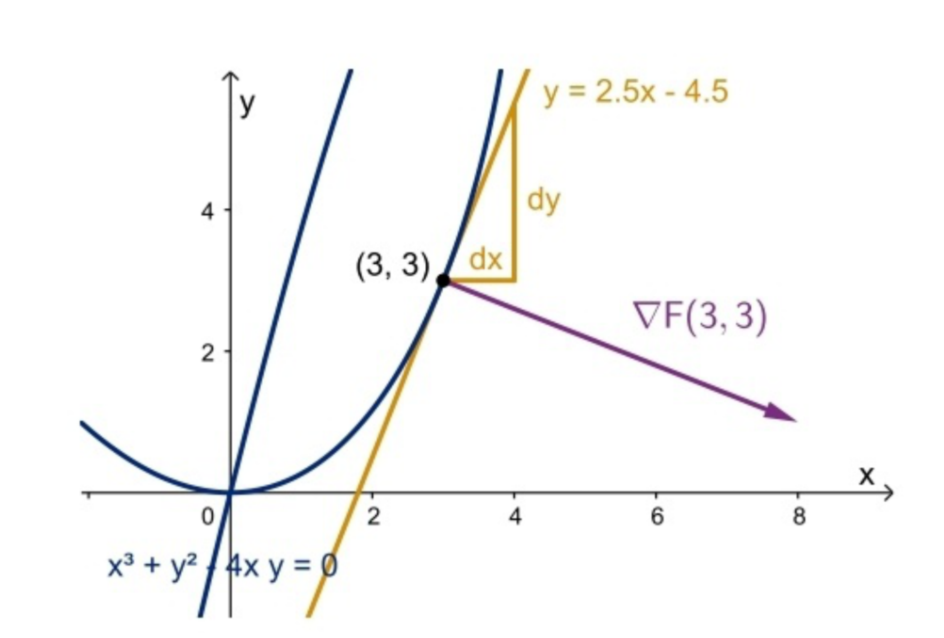
$$F_x = 3x^2 - 4y$$

$$F_y = 2y - 4x$$

so $$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{4y - 3x^2}{2y - 4x}.$$

At $(3,3)$,

$$\frac{dy}{dx} = \frac{12 - 27}{6 - 12} = \frac{5}{2}.$$



Here, $\nabla F = \langle F_x, F_y \rangle$. More on this vector later...

Exercise 1 : Compute $\frac{dy}{dx}$ where

$$x\cos(3y) + x^3y^5 = 3x - e^{xy}.$$

$\boxed{Ex}$ How would we compute $\frac{df}{d\ldots}$ where

Ex How would we compute $\frac{df}{dt}$ where

$$f(x,y,z) = \frac{x^2 - z}{y^4}$$

$$x = t^3 + 7, \quad y = \cos(2t), \quad z = 4t \ ?$$

$\boxed{\text{Chain Rule (Final Version)}}$ For a function $f(x_1, \dots, x_n)$, where each x_i, $1 \le i \le n$, is a function $x_i = x_i(t_1, \dots, t_m)$, and everything is differentiable,

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}$$

for any $1 \le j \le m$.

In the above example,

$$f_x = \frac{2x}{y^4}, \quad f_y = -4\frac{x^2-z}{y^5}, \quad f_z = \frac{-1}{y^4}$$

$$x'(t) = 3t^2, \quad y'(t) = -2\sin(2t), \quad z'(t) = 4.$$

Then

$$\frac{df}{dt} = f_x x'(t) + f_y y'(t) + f_z z'(t)$$

$$= \frac{2x}{y^4} 3t^2 + \frac{4(x^2-z)}{y^5} \cdot 2\sin(2t)$$

$$- \frac{1}{y^4} \cdot 4$$

$$= \frac{6(t^3+7)t^2}{\cos^4(2t)} + \frac{8((t^3+7)^2-4t)\sin(2t)}{\cos^5(2t)}$$

$$- \frac{4}{\cos^4(2t)}.$$

Exercise 2: An airplane is on approach to Hartsfield-Jackson Airport on an unknown trajectory, but its altitude h (in feet) is a function of its coordinates x and y on the 2-dim. radar display in the control tower (listed in miles). The plane is following some trajectory $(x(t), y(t))$, where t is in minutes, and the control tower can determine that right now,

$$\frac{\partial h}{\partial x} = -5 \text{ ft./mi.}, \quad \frac{\partial h}{\partial y} = 2 \text{ ft./mi.}$$

$$\frac{dx}{dt} = 3 \text{ mi./min.}, \quad \frac{dy}{dt} = 7 \text{ mi./min.}$$

Find the current change in the plane's altitude in ft./min.

Next time: directional derivatives.