Lecture 14.6

Last time:

- The chain rule expresses the (partial) derivative(s) of a multivariable function $f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are each (multivariable) functions of t (or more variables), as

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt}$$

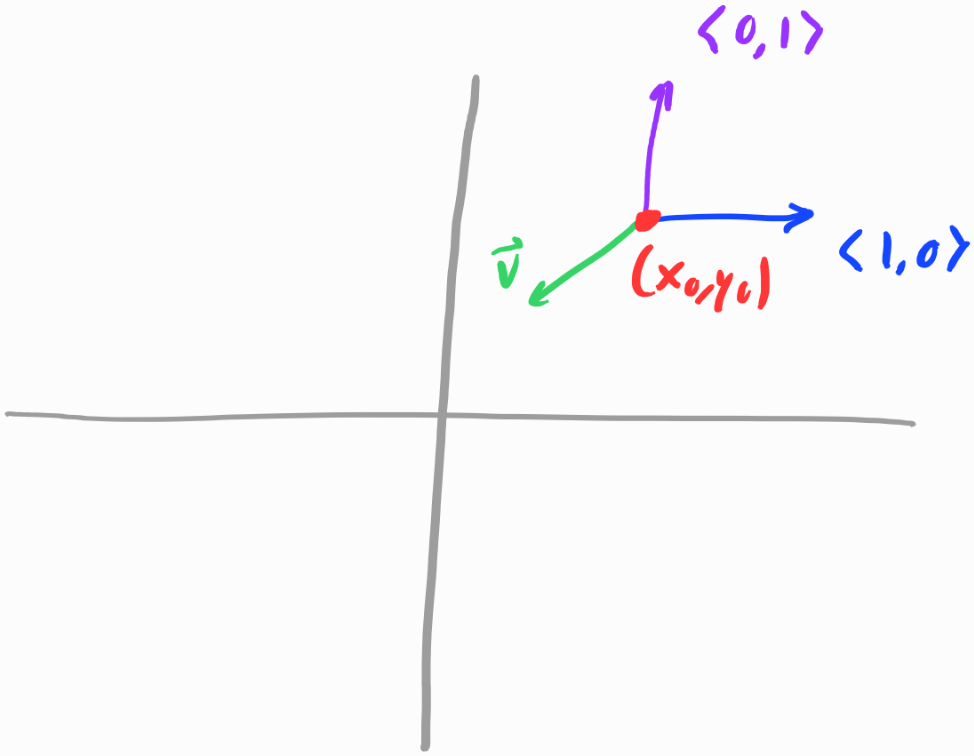
(something similar for $\frac{\partial f}{\partial t_j}$).

- For an implicit equation $F(x,y) = 0$,

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}.$$

Directional Derivatives

Partial derivatives of $f(x,y)$ measure the rates of change of f in the x- and y-directions.



But we know there are many ways

to move around in $\mathbb{R}^2$, for example along a vector $\vec{v}$ starting at (x_0, y_0).

$\boxed{\text{Def}}$ Let $f(x_1, \ldots, x_n)$ be a function and $\vec{v}$ a vector in $\mathbb{R}^n$. Then the **directional derivative** of f at a point $P = (a_1, \ldots, a_n)$ in the direction of $\vec{v}$, written $D_{\vec{v}} f(a_1, \ldots, a_n)$, is the rate of change of f at P. Explicitly, let $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ be the unit vector in the direction of $\vec{v}$. Then

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(P + h\vec{u}) - f(P)}{h}$$

$$= \lim_{h \to 0} \frac{f(a_1 + hu_1, \ldots, a_n + hu_n) - f(a_1, \ldots, a_n)}{h}.$$

Note: if $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$, then $D_{\vec{u}} f = D_{\vec{v}} f$

so we might as well just focus on unit vectors as inputs to $D_{\_} f$.

For 2-variable functions $f(x,y)$, points $P = (a,b)$ and unit vectors $\vec{u} = \langle u_1, u_2 \rangle$, this looks like:

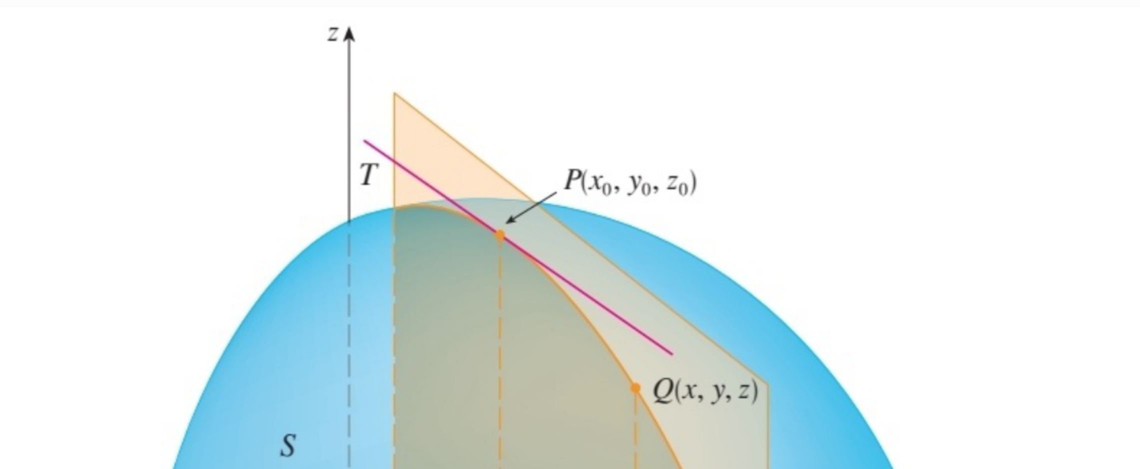
$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}.$$

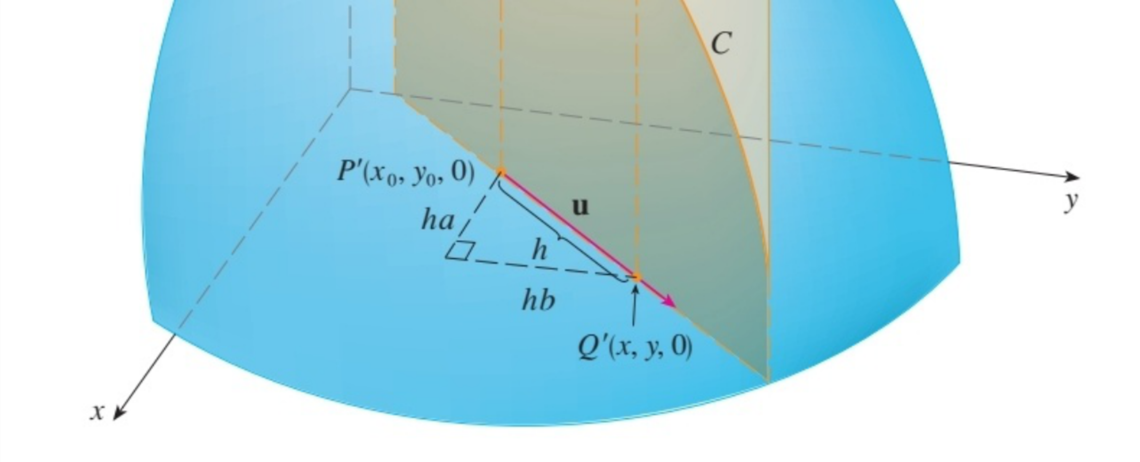
$\boxed{\text{Ex}}$ For $\vec{u} = i = \langle 1, 0 \rangle$, this is just the partial derivative with respect to x:

$$D_i f(x,y) = \frac{\partial f}{\partial x}(x,y).$$

Likewise, for $\vec{u} = j = \langle 0, 1 \rangle$,

$$D_j f(x,y) = \frac{\partial f}{\partial y}(x,y).$$



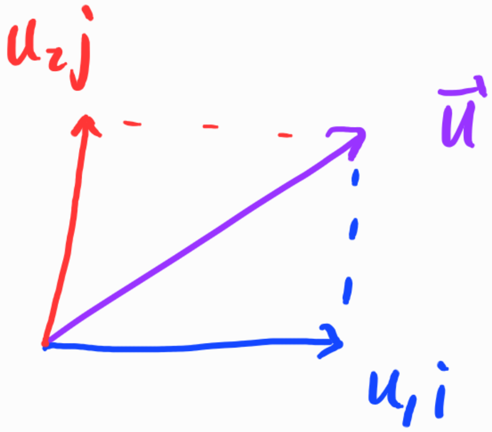


Here, you can see that the "slope" in the $\vec{u}$ direction is the slope of a cross section of $z = f(x,y)$.

Important computational tool: rather than computing $D_{\vec{u}} f$ by a limit everytime, let's decompose $\vec{u}$ into its coordinate directions. Algebraically,

$$\vec{u} = \langle u_1, u_2 \rangle = u_1 i + u_2 j.$$

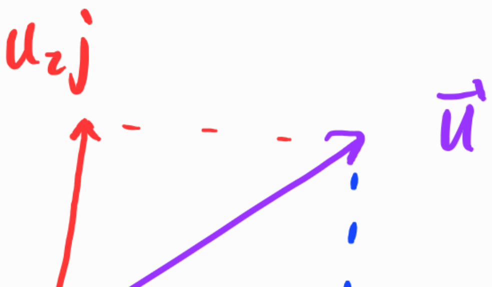
Visually,

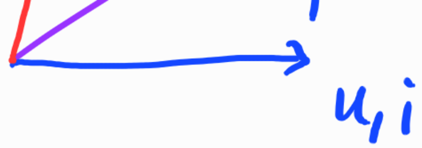


$\boxed{\text{Theorem}}$ For any function $f(x,y)$ and unit vector $\vec{u} = \langle u_1, u_2 \rangle$,

$$D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \vec{u}$$

$$= f_x u_1 + f_y u_2 .$$





change in f in $\vec{u}$ direction = change in f in x-direction + change in f in y-direction

This vector $\langle f_x, f_y \rangle$ has a special name, which we state more generally now.

$\boxed{\text{Def}}$ The gradient of $f(x_1, \ldots, x_n)$ is the vector-valued function

$$\nabla f = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle.$$

"nabla"

$\boxed{\text{Theorem}}$ For any $f(x_1, \ldots, x_n)$ and unit vector $\vec{u}$ in $\mathbb{R}^n$,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

Ex ① For $f(x,y) = x^2 + y^2$,

$$f_x = 2x \quad \text{and} \quad f_y = 2y.$$

In the direction of $\vec{v} = \langle 1, 1 \rangle$,

a unit vector is $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

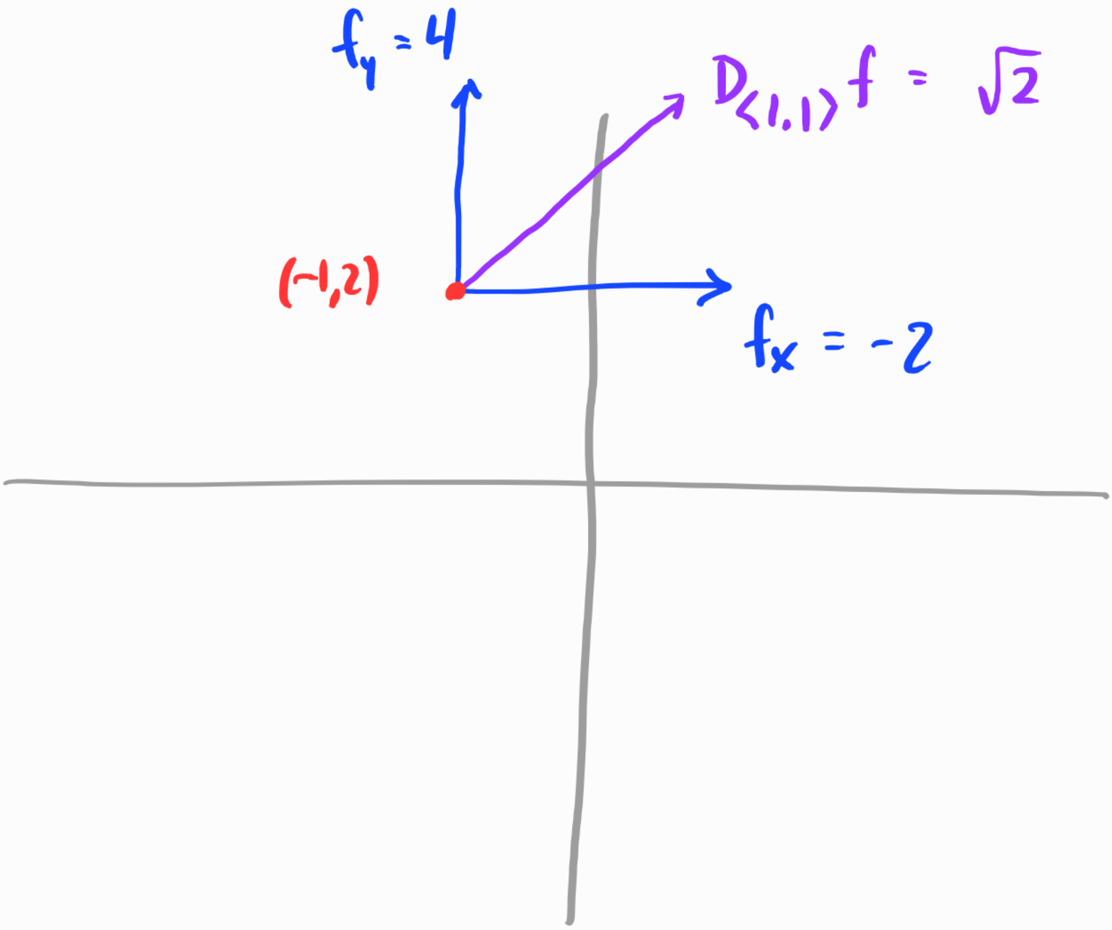
and the directional derivative is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$= \langle 2x, 2y \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$= x\sqrt{2} + y\sqrt{2}$.

Thus at $P = (-1, 2)$ for example,



Exercise 1: Compute the gradients of each function.

(a) $f = \sqrt{x^2 + y^4}$

(b) $f = x^2 \sec(3x) - \frac{x^2}{y^3}$

(c) $f = \ln(xy + z^2)$

Exercise 2: Compute $D_{\vec{v}} f$ for each of the following functions and vectors and evaluate at the point, if given.

(a) $f = x\cos y$, $\vec{v} = \langle 2,1 \rangle$, $(a,b) = (0,\pi)$

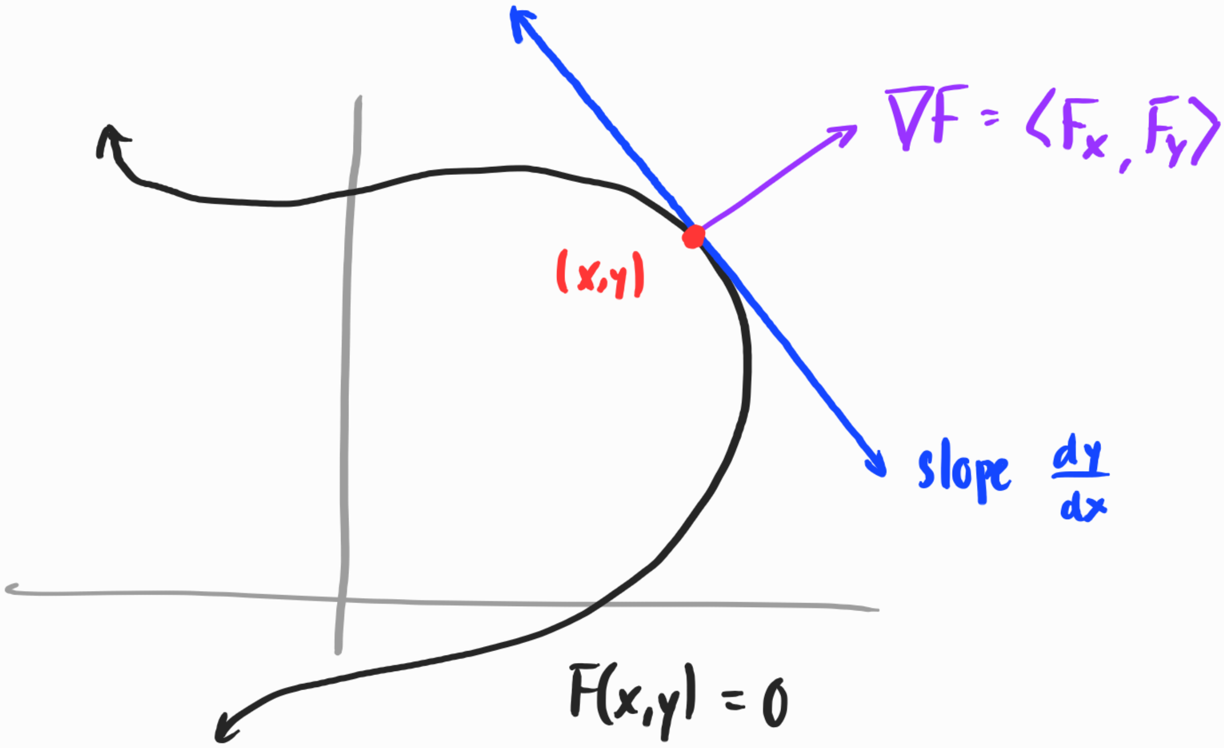
(b) $f = x^3 - 3xy + 4y^2$, $\vec{u}$ = unit vector with angle $\theta = \pi/6$, $(a,b) = (1,2)$

(c) $f = xe^{xy} + yz$, $\vec{u} = \langle -1,2,2 \rangle$

Recall that if $F(x,y) = 0$, then

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

In a picture,



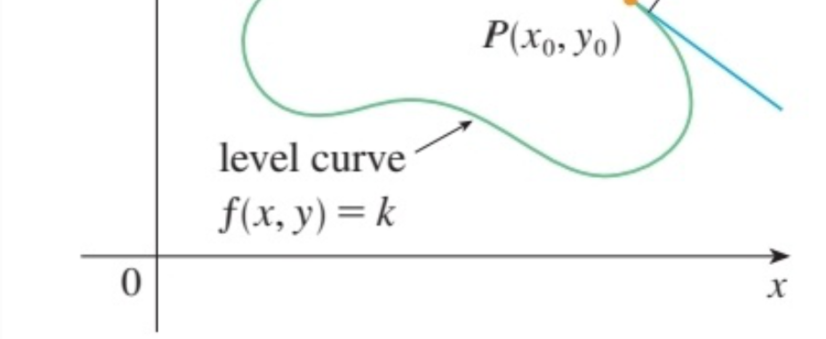
Notice that ∇F is orthogonal to the tangent line:

$$\nabla F \cdot \left\langle 1, \frac{dy}{dx} \right\rangle = \langle F_x, F_y \rangle \cdot \left\langle 1, \frac{-F_x}{F_y} \right\rangle$$

$$= F_x - F_y \frac{F_x}{F_y} = 0.$$

This is true for any level curve:

$\boxed{\text{Theorem}}$ For any function $f(x,y)$, the gradient $\nabla f(x_0,y_0)$ is orthogonal to the level curve $f(x,y) = k$ containing (x_0,y_0) (namely, $k = f(x_0,y_0)$).



What does this mean in terms of rates of change of f?

Well, $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ can be rewritten

$$D_{\vec{u}} f = |\nabla f| \cos(\theta)$$

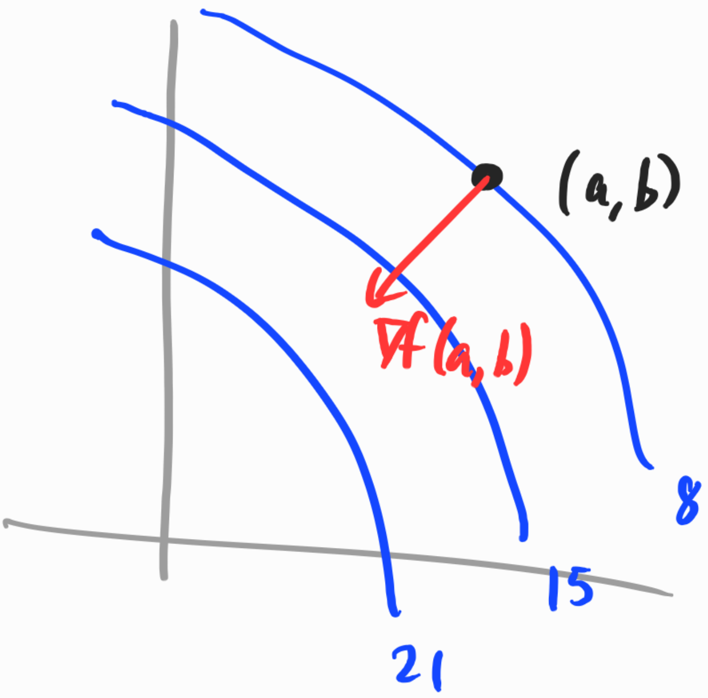
where θ is the angle between ∇f and $\vec{u}$.

This is maximized when $\theta = 0$, i.e.

when $\vec{u}$ is in the same direction as ∇f.

Similarly, $D_{\vec{u}} f$ is minimized when $\theta = \pi$, i.e. when $\vec{u}$ is in the opposite direction as ∇f.

Visually, "the fastest rate of change is at a right angle to the level curves":



Exercise 3: Find the maximum rate of change of $f(x,y) = \sqrt{x^2 + y^4}$ at $(-2, 3)$. Which direction does this maximum rate occur in?

Exercise 4: The elevation of a hill is given by the height function

$$h(x,y) = 1000 - \frac{1}{100}x^2 - \frac{1}{50}y^2.$$

Standing at the point $(60, 100)$,

(a) in what direction is the elevation changing fastest?

(b) what is this elevation change?

Next time: optimization 👻