Lecture 14.7, part 2

Last time:

- We defined local and absolute minimum and maximum values for $f(x_1, \ldots, x_n)$.

- If the partial derivatives of f exist at a max./min., they are all 0 at the point, i.e. the point is a critical point.

- To classify critical points of $f(x,y)$, compute the discriminant

$$D = f_{xx} f_{yy} - f_{xy} f_{yx}$$

$= f_{xx} f_{yy} - f_{xy}^2$ when the partials are continuous

and follow these rules:

* $D > 0$, $f_{xx} > 0 \Rightarrow$ local min.
* $D > 0$, $f_{xx} < 0 \Rightarrow$ local max.
* $D < 0 \Rightarrow$ saddle pt.

Recall: the extreme value theorem (EVT) in Calc 1 says any continuous function on a closed interval has a max.

and a min. value on that interval.

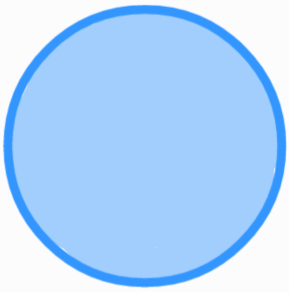
In multivariable land, we have:

Extreme Value Theorem For a function $f(x_1, \dots, x_n)$ on a closed and bounded region D in $\mathbb{R}^n$, f has both a maximum and a minimum value in D.

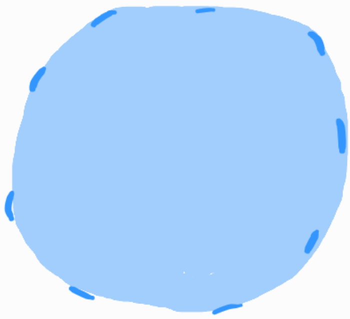
Here, **bounded** means the points in

D are all no further than a fixed distance from each other.

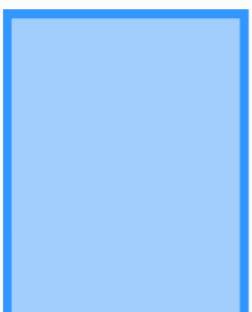
Closed means the points on the boundary of D are included.

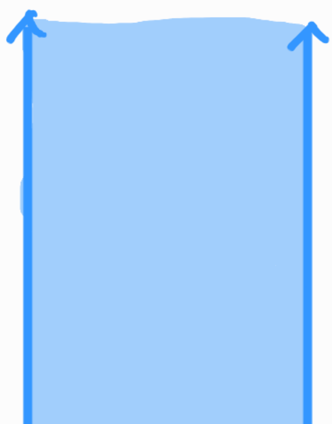


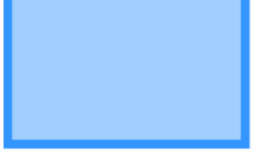
closed + bounded



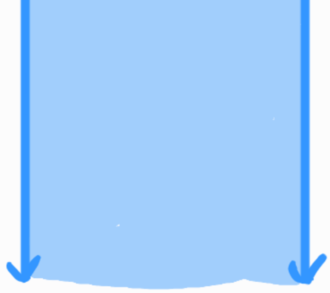
bounded but not closed







closed + bounded

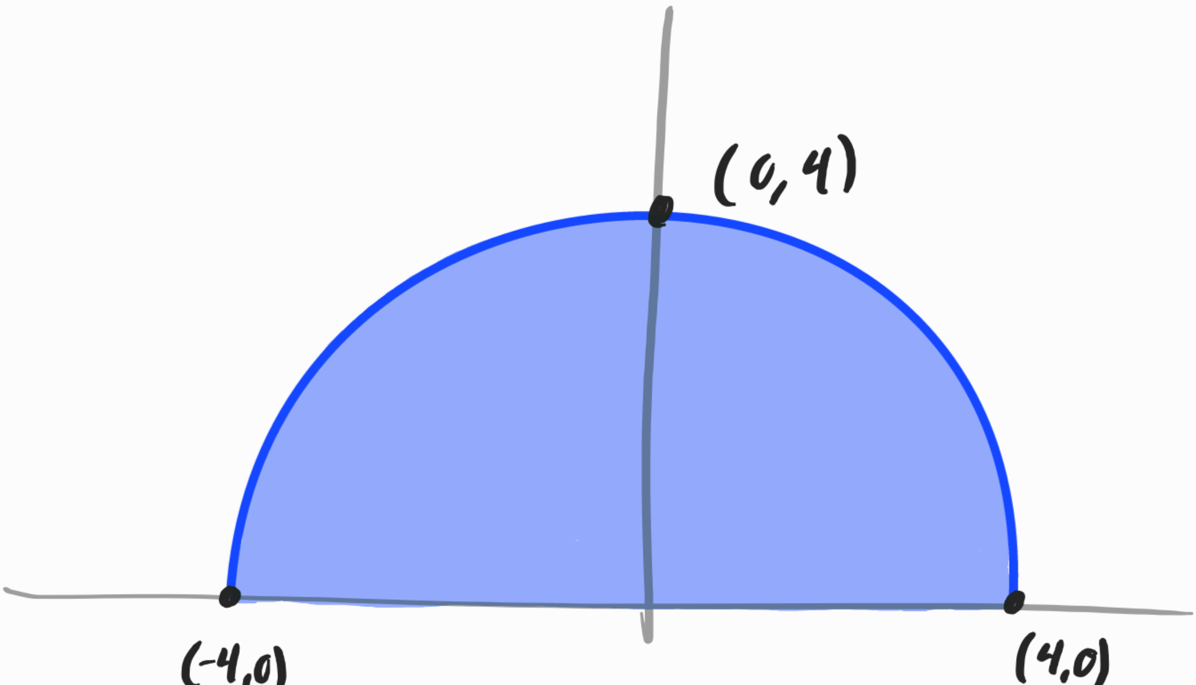


closed but not bounded

$\boxed{\text{Ex}}$ Let's find the extreme values of

$$f(x,y) = x^2 + 2y^2 - x^2y$$

on the region



D can be written explicitly as

$$D = \{(x,y) \mid x^2+y^2 \leq 16,\ y \geq 0\}.$$

This is closed and bounded (why?)

so by the EVT, f has a max. and a min. somewhere on f, either on the interior at a critical pt. of f or on the boundary of D.

Interior: the critical pts. are found by

$$f_x = 2x - 2xy = 0$$

$$f_y = 4y - x^2 = 0$$

$$(1-y)2x = 0 \Rightarrow x = 0 \text{ or } y = 1$$

$\underline{x=0}$: $4y = 0 \Rightarrow y = 0$

$\underline{y=1}$: $4 - x^2 \Rightarrow x = \pm 2.$

Our list of critical pts. is

$$(0,0),\ (-2,1),\ (2,1)$$

all of which are on the interior

of D.

<u>Boundary</u>: all points on the boundary satisfy $x^2+y^2=16$ (or $y=0$).

Substituting this into f, we get

$$f = x^2+2y^2-x^2y = \underbrace{x^2+y^2}_{=16}+y^2-x^2y$$

$$= y^2-\underbrace{x^2}_{=16-y^2}y+16$$

$$= y^2-(16-y^2)y+16 = y^3+y^2-16y+16.$$

This is a single variable function and the admissible y-values on the boundary are $0 \le y \le 4$.

By Calc 1 techniques:

$$f'(y) = \underbrace{3y^2 + 2y - 16}_{(3y+8)(y-2)} = 0$$

$$\Longrightarrow \quad y = \cancel{-8/3},\ 2$$

not in interval

At $y = 2$, there are two solutions to

$$x^2 + 4 = 16,$$

namely $x = \pm 2\sqrt{3}$.

The endpoints of the interval $0 \leq y \leq 4$

also contribute pts. $(\pm 4, 0)$, $(0, 4)$.

Finally, on the bottom segment

$y = 0$, $-4 \leq x \leq 4$, the function

is $f = x^2$, so

$$f'(x) = 2x \implies x = 0.$$

But $(0,0)$ and the endpoints $(\pm 4, 0)$

are already on the list.

Extrema: Here's a table of all our pts. so far, together with their $f(x,y)$-values:

(x,y)	$f(x,y)$
$(0,0)$	0
$(-2,1)$	2
$(2,1)$	2
$(-2\sqrt{3},2)$	-4
$(2\sqrt{3},2)$	-4
$(-4,0)$	16
$(4,0)$	16

MIN

$(0,4)$ | 32 MAX

This shows f has its max. value on D at $(0,4)$ and its min. value on D at both $(\pm 2\sqrt{3}, 2)$.

Exercise 1: Find the absolute min. and max. values of

$$f(x,y) = 2x^2 - y^2 + 6y$$

on $D = \{(x,y) \mid x^2 + y^2 \leq 16\}$.

Exercise 2: Find the absolute min. and max. values of

$$f(x,y) = 2x^3 - 4y^3 + 24xy$$

on $D = \{(x,y) \mid 0 \leq x \leq 5, -3 \leq y \leq -1\}$.

Exercise 3: Find the absolute min. and max. values of

$$f(x,y) = 18x^2 + 14y^2 - y^2x - 2$$

on the solid triangle with vertices

$(-1,-1)$, $(5,-1)$ and $(5,17)$.