Lecture 14.7

Last time:

- The directional derivative

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

measures the rate of change of f in the direction of any unit vector $\vec{u}$.

- The gradient of f is

$$\nabla f = \langle f_{x_1}, \dots, f_{x_n} \rangle.$$

- $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

- In particular, f changes fastest in the direction of ∇f.

$\boxed{\text{Optimization}}$

Motivation: what are the largest and smallest values of f, possibly subject to some constraints.

$\boxed{\text{Def}}$ A function $f(x_1, \ldots, x_n)$ has a local minimum value at a point $P = (a_1, \ldots, a_n)$ if, for all $(x_1, \ldots, x_n)$ near P (say, within a small sphere with center P),

$$f(x_1, \ldots, x_n) \geq f(a_1, \ldots, a_n).$$

An local maximum value of f is defined similarly, with

$$f(x_1, \ldots, x_n) \leq f(a_1, \ldots, a_n)$$

instead. A global or absolute maximum/minimum value is one maximum/minimum value is the

for which the appropriate inequality holds for all $(x_1, \dots, x_n)$ in the domain of f.

Theorem If f has a local maximum or minimum at $P = (a_1, \dots, a_n)$ and $\frac{\partial f}{\partial x_i}$ is defined at P for all $1 \leq i \leq n$, then

$$\frac{\partial f}{\partial x_1}(a_1, \dots, a_n) = \cdots = \frac{\partial f}{\partial x_n}(a_1, \dots, a_n) = 0.$$

max & min

$\boxed{\text{Ex}}$ Let's find the extrema of

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14.$$

Just like in Calc 1, we do this by searching for critical points.

Here, these are points where both

$$f_x = 0 \quad \text{and} \quad f_y = 0.$$

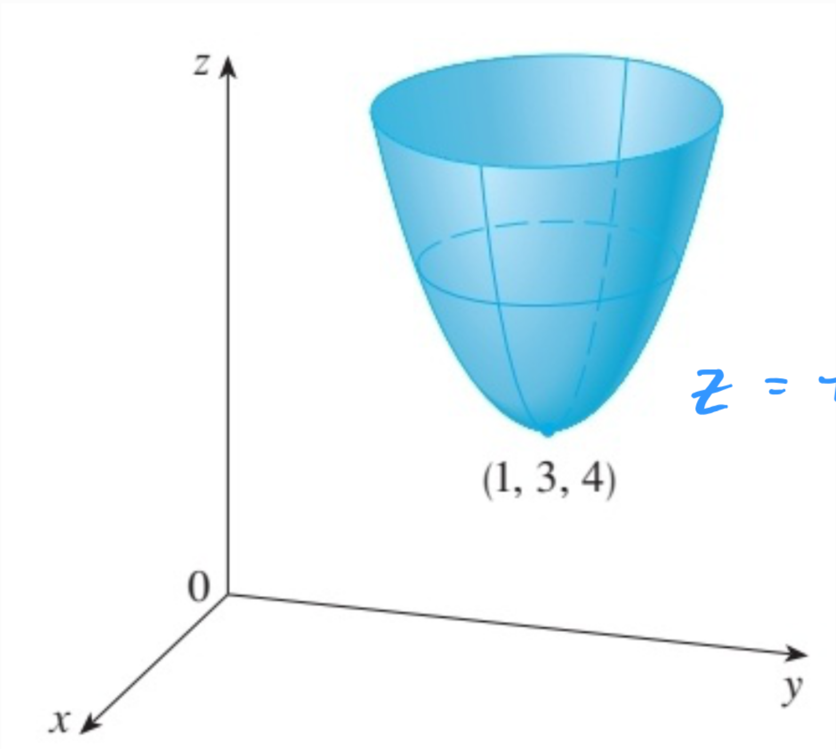
We have:

$$f_x = 2x - 2, \quad f_y = 2y - 6.$$

Then $f_x = 0$ when $x = 1$

$f_y = 0$ when $y = 3$,

So $(x, y) = (1, 3)$ is the only critical point of f.



$z = f(x, y)$

Looking at the figure, we can see that f has an absolute minimum at $(1,3)$.

Can we "see" this algebraically?

Notice that

$$f = x^2 - 2x + y^2 - 6y + 14$$

$$= (x-1)^2 + (y-3)^2 + 4.$$

Then $f(1,3) = 4$ is easy to spot.

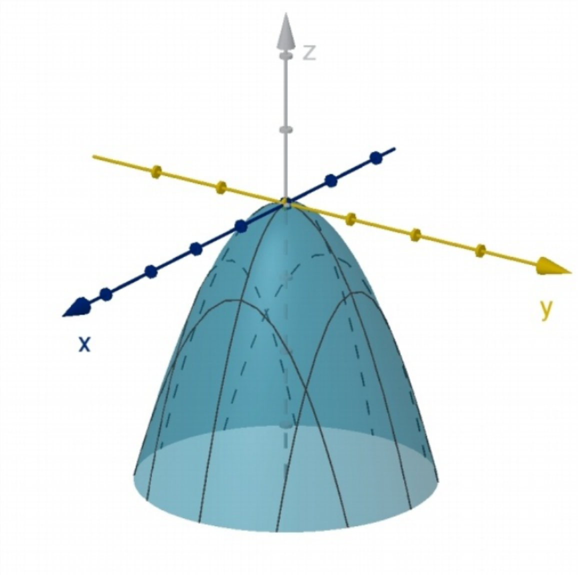
For any (x,y),

$$(x-1)^2 + (y-3)^2 + 4 \geq 4$$

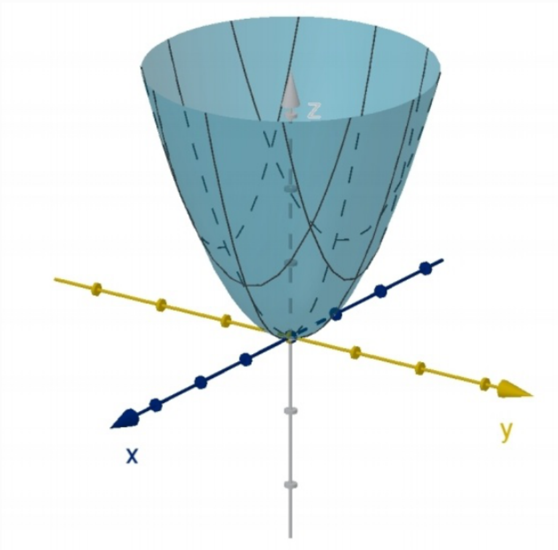
so $(1,3)$ does in fact give a minimum value.

Exercise 1: Explain why $(1,3)$ is the only point where f has a minimum. Does f have a maximum?

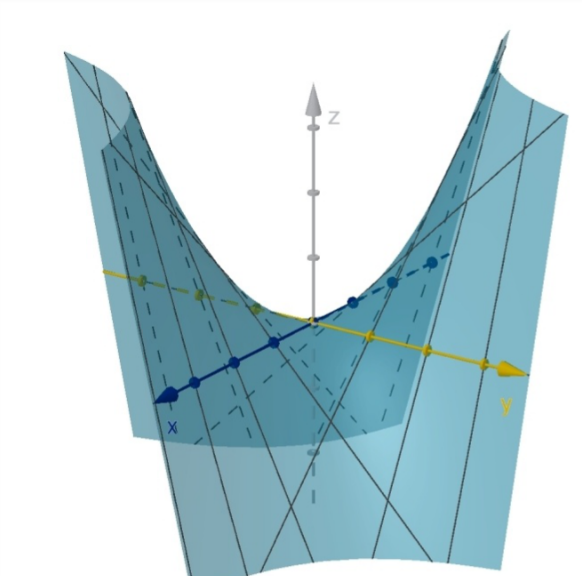
Warning: at a critical point, a function $f(x,y)$ could have one of three shapes :



a local max.



a local min.



a saddle point

To detect these, we will use the two-variable version of the second derivative test from Calc 1:

Theorem For a function $f(x,y)$ with a critical point at (a,b) and all second order partial derivatives continuous near (a,b), define

$$D = f_{xx} f_{yy} - \underbrace{f_{xy} f_{yx}}_{= f_{xy}^2 \text{ by the}} ,$$

mixed partials theorem

Then: (1) If $D > 0$ and $f_{xx} > 0$ at (a,b), f has a local minimum there.

(2) If $D > 0$ and $f_{xx} < 0$ at (a,b), f has a local maximum there.

(3) If $D < 0$ at (a,b), f has a saddle point at (a,b).

Otherwise, the test is inconclusive.

Note: if $D > 0$, f_{xx} and f_{yy}

have the same sign.

$\boxed{\text{Ex}}$ For $f(x,y) = \cos(2x+y) + xy$,

we have :

$$f_x = -2\sin(2x+y) + y$$

$$f_y = -\sin(2x+y) + x$$

$$f_{xx} = -4\cos(2x+y)$$

$$f_{yy} = -\cos(2x+y)$$

$$f_{xy} = -2\cos(2x+y) + 1.$$

Notice that $(0,0)$ is a critical

point (there are other!) and

$$f_{xx}(0,0) = -4$$

$$f_{yy}(0,0) = -1$$

$$f_{xy}(0,0) = -1.$$

Then $D(0,0) = f_{xx} f_{yy} - f_{xy}^2$

$$= (-4)(-1) - (-1)^2$$

$$= 4 - 1 = 3 > 0.$$

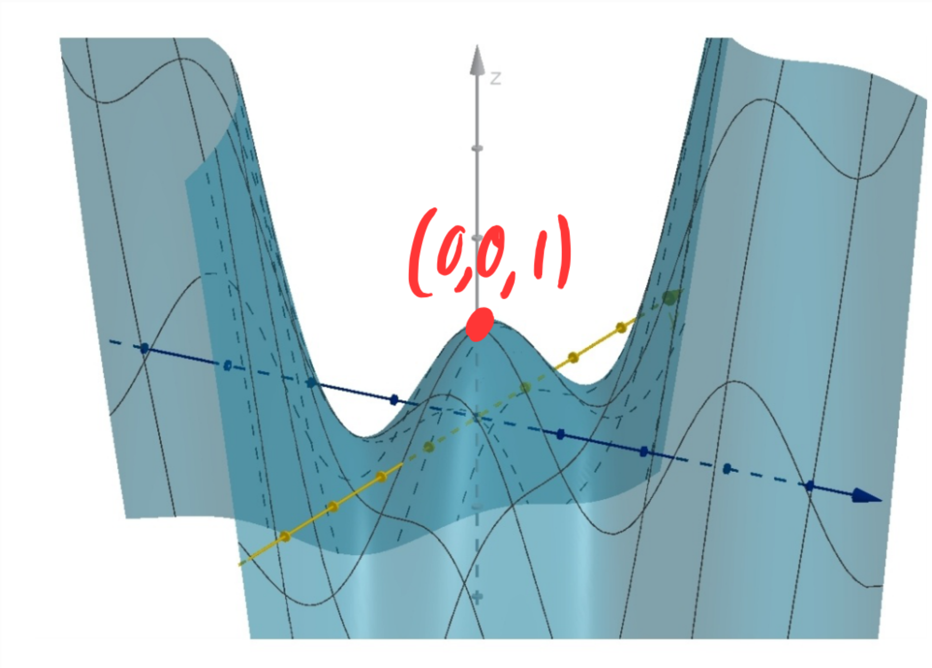
This shows f has a max. or min.

at $(0,0)$.

Further, since $f_{xx}(0,0) = -4 < 0$

Further, since $f_{xx}(0,0) = -4 < 0$,

f has a local max. at $(0,0)$.



$z = \cos(2x+y) + xy$

Exercise 2: Find and classify all critical points of the functions.

(a) $f = (y-2)x^2 - y^2$

(b) $f = 2x^2 - y^2 + 6y$

(c) $f = 2x^3 - 4y^3 + 24xy$

Exercise 3: After decades of research, the Lucky Tails Saddle Company has perfected their saddle design, which can be modeled by the graph of

$$f(x,y) = \frac{4}{5}x^2 - \frac{9}{10}y^2$$

over the domain $\{-2 \le x \le 3,\ -2 \le y \le 2\}$.

Find the saddle point of this saddle.

Recall: the extreme value theorem (EVT) in Calc 1 says any continuous function on a closed interval has a max. and a min. value on that interval.

In multivariable land, we have:

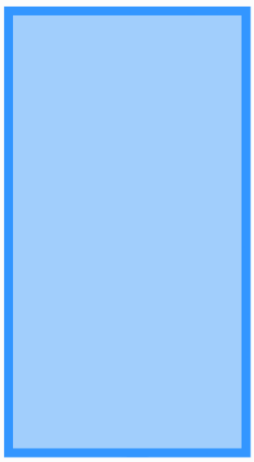
$\boxed{\text{Extreme Value Theorem}}$ For a function $f(x_1, \ldots, x_n)$ on a closed and bounded region D in $\mathbb{R}^n$, f has both a maximum and a minimum value in D.

Here, bounded means the points in D are all no further than a fixed distance from each other.

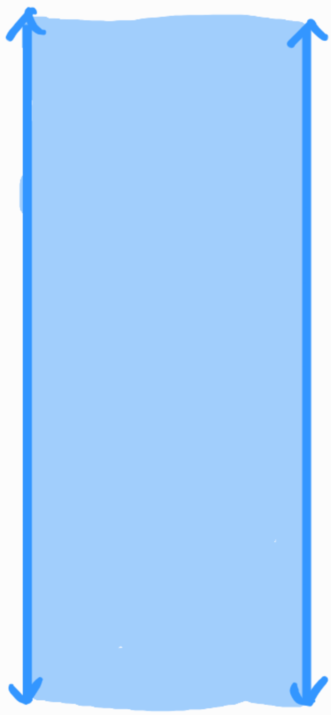
Closed means the points on the boundary of D are included.

closed + bounded

bounded but not closed



closed + bounded

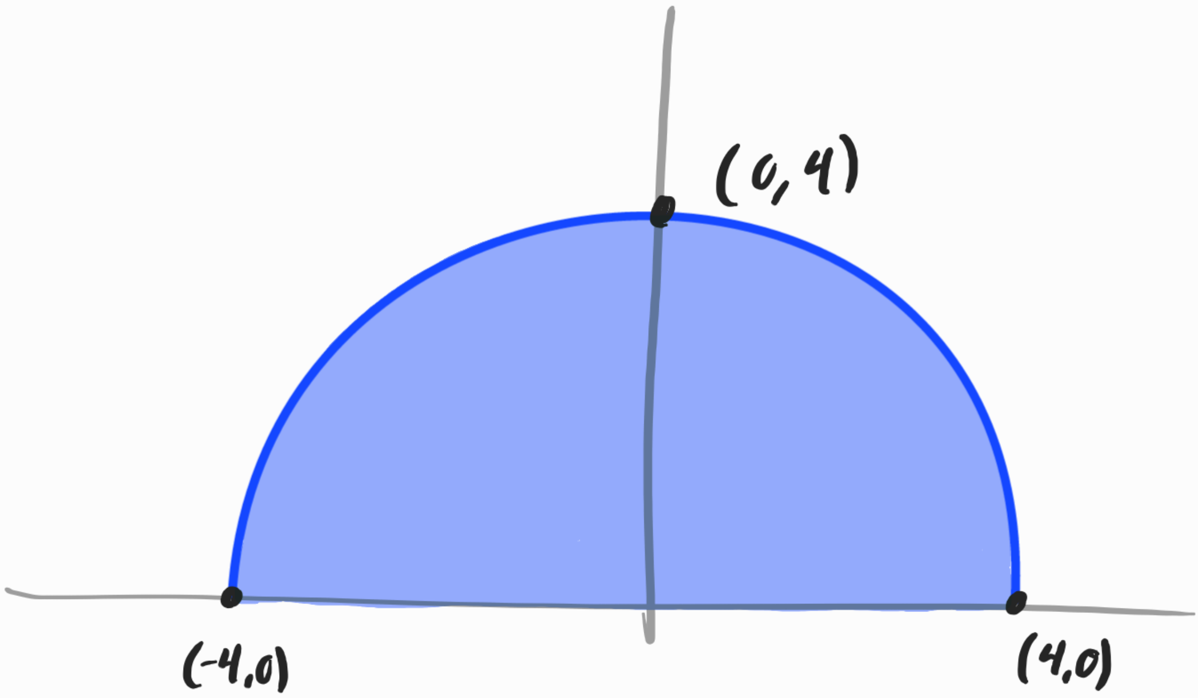


closed but not bounded

$\boxed{\text{Ex}}$ Let's find the extreme values of

$$f(x,y) = x^2 + 2y^2 - x^2y$$

on the region



D can be written explicitly as

$$D = \{(x,y) \mid x^2 + y^2 \leq 16, \ y \geq 0\}.$$

This is closed and bounded (why?)

so by the EVT, f has a

max and a min somewhere on

max. and a min. somewhere on

f, either on the interior at a critical pt. of f or on the boundary of D.

<u>Interior</u> : the critical pts. are found by

$$f_x = 2x - 2xy = 0$$

$$f_y = 4y - x^2 = 0$$

$$(1-y)2x = 0 \implies x = 0 \text{ or } y = 1$$

<u>$x=0$</u> : $4y = 0 \implies y = 0$

<u>$y=1$</u> : $4 - x^2 \Rightarrow x = \pm 2$

Our list of critical pts. is

$$(0,0),\ (-2,1),\ (2,1)$$

all of which are on the interior of D.

<u>Boundary</u> : all points on the boundary satisfy $x^2+y^2=16$ (or $y=0$).

Substituting this into f, we get

$$f = x^2+2y^2-x^2y = \underbrace{x^2+y^2}_{=16}+y^2-x^2y$$

$$= y^2 - \underbrace{x^2}_{=16-y^2} y + 16$$

$$= y^2 - (16 - y^2)y + 16 = y^3 + y^2 - 16y + 16.$$

This is a single variable function and the admissible y-values on the boundary are $0 \le y \le 4$.

By Calc 1 techniques:

$$f'(y) = \underbrace{3y^2 + 2y - 16}_{(3y+8)(y-2)} = 0$$

$$\Longrightarrow \quad y = \cancel{-8/3},\ 2$$

not in interval

At $y=2$, there are two solutions to

$$x^2 + 4 = 16,$$

namely $x = \pm 2\sqrt{3}$.

The endpoints of the interval $0 \le y \le 4$ also contribute pts. $(\pm 4, 0)$, $(0, 4)$.

Finally, on the bottom segment $y = 0$, $-4 \le x \le 4$, the function is $f = x^2$, so

$$f'(x) = 2x \implies x = 0.$$

But $(0,0)$ and the endpoints $(\pm 4, 0)$ are already on the list.

<u>Extrema</u>: Here's a table of all our pts. so far, together with their $f(x,y)$-values:

(x,y)	$f(x,y)$
$(0,0)$	0
$(-2,1)$	2
$(2,1)$	2

$(-2\sqrt{3}, 2)$	-4
$(2\sqrt{3}, 2)$	-4
$(-4, 0)$	16
$(4, 0)$	16
$(0, 4)$	32

MIN

MAX

This shows f has its max. value on D at $(0,4)$ and its min. value on D at both $(\pm 2\sqrt{3}, 2)$.

Next time: more optimization.