Lecture 14.8

Last time:

- A continuous function on a closed, bounded region has a minimum and a maximum value.

Constrained Optimization

Optimizing a function on a closed, bounded region is an example of constrained optimization:

maximize subject to (equations

$f(x_1, \dots, x_n)$ (or inequalities)

$\boxed{\text{Theorem (Lagrange Multipliers)}}$ The minimum and maximum values of $f(x_1, \dots, x_n)$ subject to the condition

$$g(x_1, \dots, x_n) = k,$$

a real number

where f and g are differentiable, occur where the gradient vectors ∇f and ∇g are parallel. That is, where

$$\nabla f = \lambda \nabla g$$

for some number λ

for some number λ.

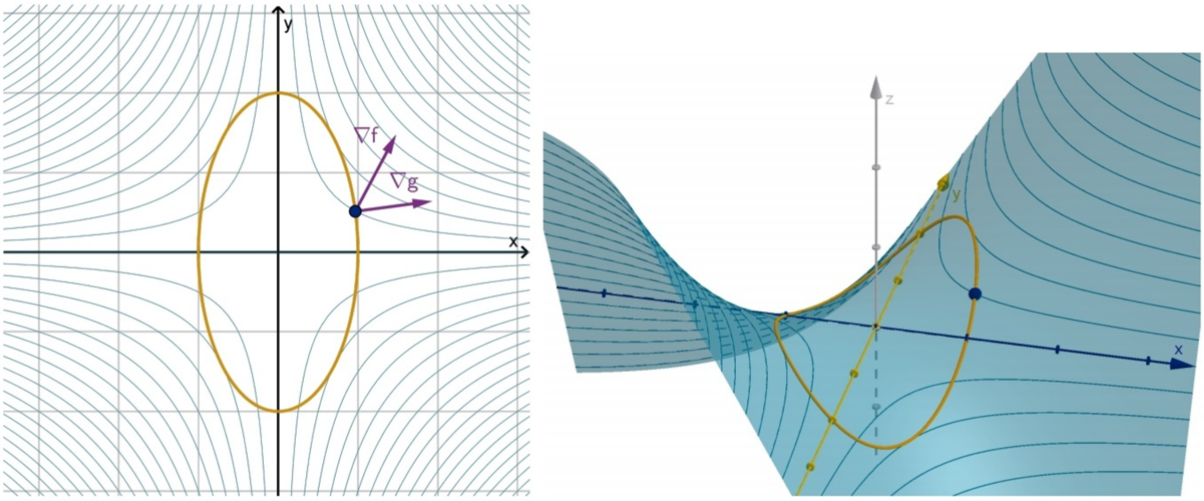


Figure: Where ∇f is not parallel to ∇g, we can travel along $g(x, y) = c$ and increase the value of f. This is because $D_{\vec{u}} f > 0$ for some $\vec{u}$ along the constraint.

Idea (in $\mathbb{R}^2$): ∇f is orthogonal to tangent vectors to the curve $g = k$, but this is a level curve of g, so ∇g is orthogonal too.

$\boxed{\text{Ex}}$ Maximize $f(x,y) = xy$ on the curve

$$4x^2 + y^2 = 4.$$

Let $g(x,y) = 4x^2 + y^2$.

Then we are looking to maximize f subject to $g = 4$. This will occur at some point satisfying

$$\nabla f = \lambda \nabla g.$$

We have

$$\nabla f = \langle y, x \rangle, \quad \nabla g = \langle 8x, 2y \rangle$$

so $\langle y, x \rangle = \lambda \langle 8x, 2y \rangle$ becomes two equations:

$$y = 8\lambda x$$
$$x = 2\lambda y.$$

Substituting $y = 8\lambda x$ into the second equation, we get

$$x = 2\lambda(8\lambda x) = 16\lambda^2 x$$

$$\Rightarrow (16\lambda^2 - 1)x = 0$$

$$16\lambda^2 = 1 \qquad \boxed{\begin{matrix} x = 0 \\ y = 0 \end{matrix}}$$

$$\lambda = \pm$$

$$y = \pm 2x \qquad 4x^2 + y^2 = 4$$

$$4x^2 + 4x^2 = 4$$

$$2x^2 = 1$$

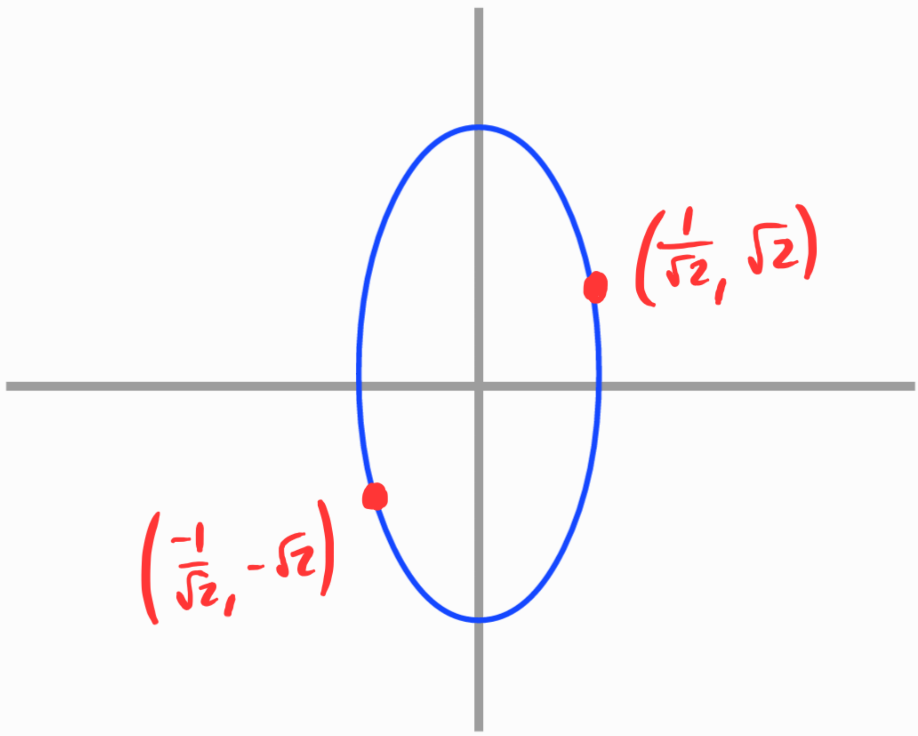
$$\boxed{\begin{aligned} x &= \pm \frac{1}{\sqrt{2}} \\ y &= \pm \sqrt{2} \end{aligned}}$$

Now we test our points to see where the maximum value occurs:

(x,y)	$f(x,y) = xy$
$(0,0)$	0
$\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$	1
$\left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$	-1

[illegible]	
$\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right)$	-1
$\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$	1

So $f(x,y) = xy$ has a maximum value of 1 at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$ on the ellipse $4x^2 + y^2 = 4$.



Note: This gives an alternative method for identifying critical points on the boundary in an Extreme Value Theorem problem.

Exercise 1: Find the minimum and maximum values of

$$f(x,y) = x^2 + 2y^2 - x^2y$$

on the region

$$D = \{(x,y) \mid x^2+y^2 \leq 16,\ x \geq 0\}.$$

$\boxed{\text{Ex}}$ Find the maximum value of

$$f(x,y,z) = x^4y^4z \quad \text{on the sphere}$$

$$x^2+y^2+z^2 = 36.$$

Here, the constraint can be written

as $g = 36$ where

$$g(x,y,z) = x^2+y^2+z^2.$$

The extrema must occur where

$$\nabla f = \lambda \nabla g.$$

We have

$$\nabla f = \langle 4x^3y^4z,\ 4x^4y^3z,\ x^4y^4 \rangle$$

$$\nabla g = \langle 2x,\ 2y,\ 2z \rangle.$$

Then $\nabla f = \lambda \nabla g$ becomes the system

$$4x^3y^4z = 2\lambda x$$

$$4x^4y^3z = 2\lambda y$$

$$x^4y^4 = 2\lambda z.$$

First, the points $(0, y, 0)$ and $(x, 0, 0)$ satisfy these equations, and the constraint $x^2+y^2+z^2 = 36$ in these

cases gives us

$(0,y,0)$ where $y = \pm 6$

$(x,0,0)$ where $x = \pm 6$.

If x, y, z are all nonzero, we can

multiply each equation by them to get

a common term:

$$yz \cdot 4x^3y^4z = 2\lambda x \cdot yz$$

$$xz \cdot 4x^4y^3z = 2\lambda y \cdot xz$$

$$xy \cdot x^4y^4 = 2\lambda z \cdot xy$$

all equal

Setting the first two new expressions equal to each other gives us

$$4x^3y^5z^2 = 4x^5y^3z^2$$

$$\Longrightarrow \underbrace{4x^3y^3z^2(x^2-y^2)}_{} = 0$$

$y = x$ or $y = -x$

On the other hand, the first and third equations are also equal, so

$$4x^3y^5z^2 = x^5y^5$$

$$\Longrightarrow x^3y^5(4z^2 - x^2) = 0$$

$x = 2z$ or $x = -2z$

In the constraint $x^2+y^2+z^2 = 36$, we can substitute this information:

$$(\pm 2z)^2 + (\pm 2z)^2 + z^2 = 36$$

$$\implies 9z^2 = 36$$

$$\implies z^2 = 4$$

$$\implies z = 2 \text{ or } -2.$$

The points satisfying the Lagrange formula $\nabla f = \lambda \nabla g$ are therefore

$(\pm 6, 0, 0)$, $(0, \pm 6, 0)$, $(\pm 4, \pm 4, \pm 2)$.

Their $f(x,y,z)$ values are:

(x,y,z)	$f(x,y,z) = x^4 y^4 z$
$(\pm 6, 0, 0)$	0
$(0, \pm 6, 0)$	0
$(\pm 4, \pm 4, 2)$	$4^8 \cdot 2 = 2^{17}$
$(\pm 4, \pm 4, -2)$	$4^8 \cdot (-2) = -2^{17}$

The max. value is $2^{17} = 131072$.

Theorem If $f(x_1, \dots, x_n)$ is subject to two (or more) constraints, say

$$g(x_1, \dots, x_n) = c$$

and $$h(x_1, \dots, x_n) = d$$

real numbers

then any minimum and maximum values of f satisfying these equations occur where ∇f is a linear combination of ∇g and ∇h:

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

Exercise 2: Show $(3,3)$ is not a local maximum of

$$f(x,y) = 2x^2 - 4xy + y^2 - 8y$$

along the curve $x^3 + y^3 = 6xy$.

Exercise 3: Find the minimum and maximum values of

$$f(x,y) = x^2 + 6xy + 9y^2 + 5$$

in the region

$$D = \{(x,y) \mid x^2 + y^2 \le 10\}$$

$D = \{ (x,y) \mid \quad 0 \le x \le y \}.$

Exercise 4: Find the maximum value of $f(x,y,z) = xy^2z^3$ subject to the constraints that $x, y, z \ge 0$ and $x + y + z = 1$.

Exercise 5: Find the maximum of $f(x,y,z) = x(y+z)$ subject to $x^2 + y^2 = 1$ and $xz = 1$.

Next time: double integrals.