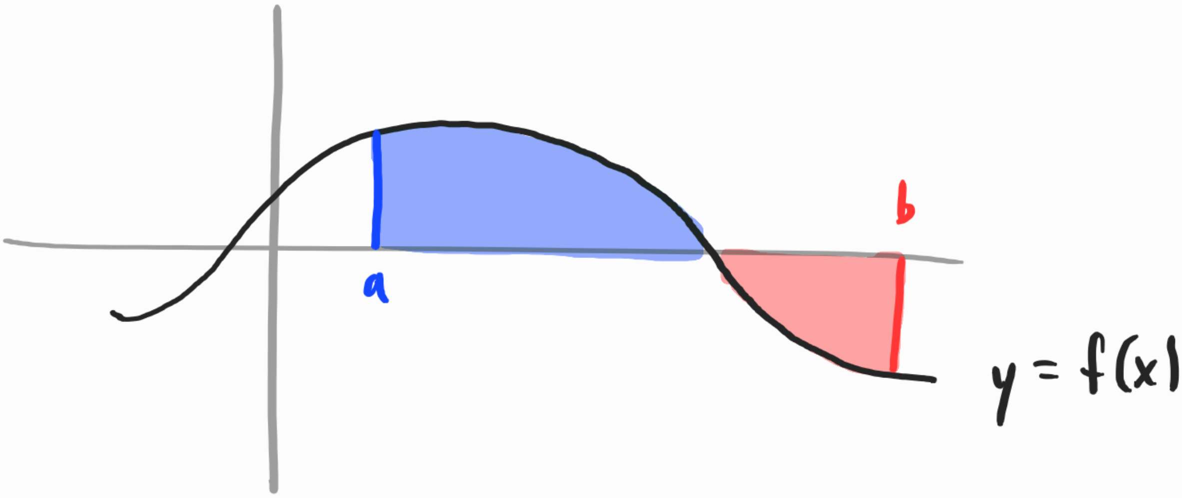
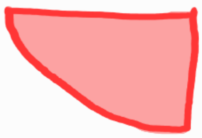
Lecture 15.1

Double Integrals

Recall: the definite integral of $f(x)$ over $[a, b]$

is the signed area



$$\int_a^b f(x)\,dx =$$  $-$ 

\# of rectangles

$$= \lim_{n \to \infty} \sum^{n} f(x_i^*)\Delta x$$

$i=1$

height of ith rectangle

base of each rectangle

For a function $f(x,y)$, we can capture the <u>volume</u> under the graph $z = f(x,y)$ in a similar way:

$\boxed{\text{Def}}$ For a function $f(x,y) \geq 0$ on a rectangular region

$$R = [a,b] \times [c,d] = \left\{ (x,y) \;\middle|\; \begin{matrix} a \leq x \leq b, \\ c \leq y \leq d \end{matrix} \right\},$$

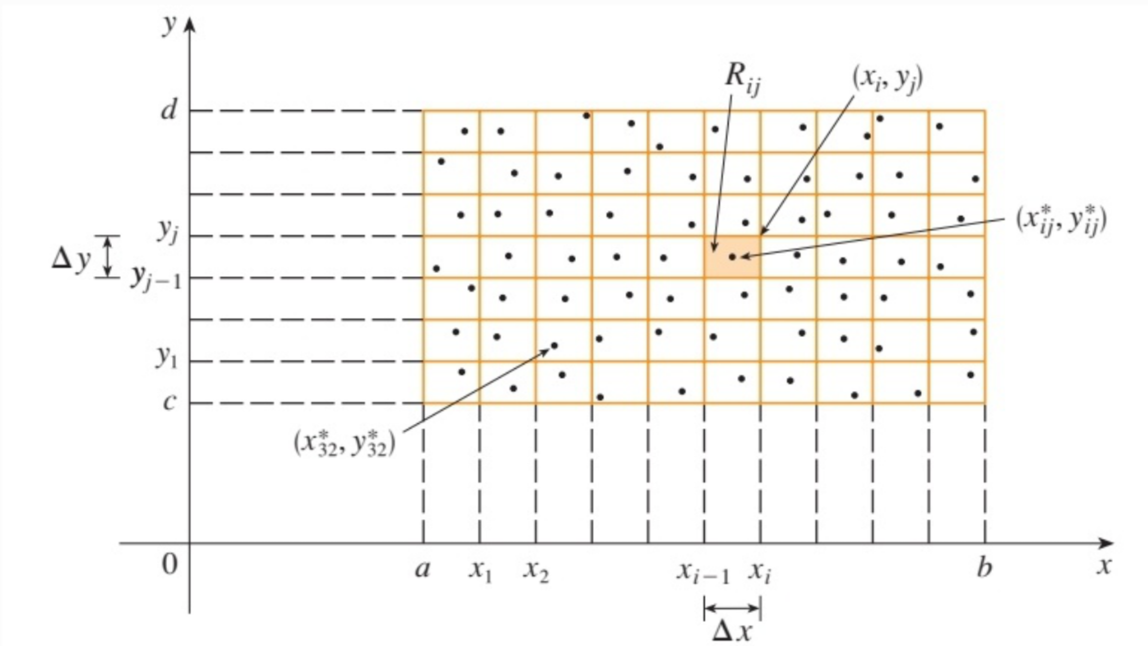
the **double integral** of f over R is the volume under the graph $z = f(x,y)$ over R, denoted

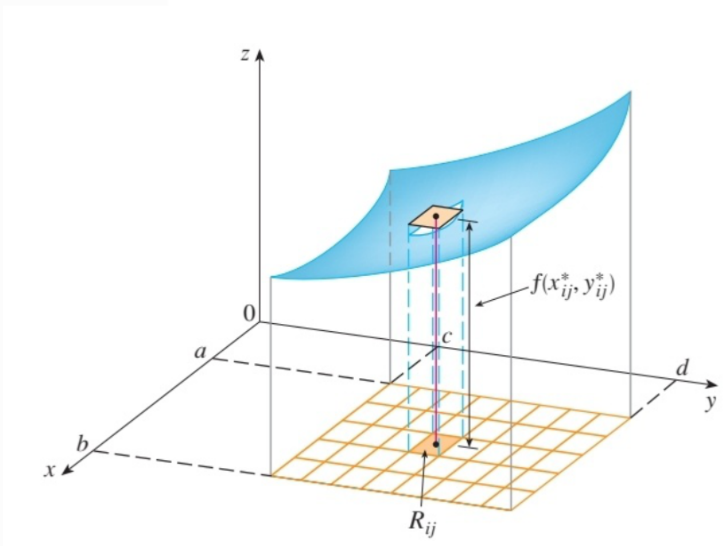
$$\iint_R f(x,y)\, dA.$$

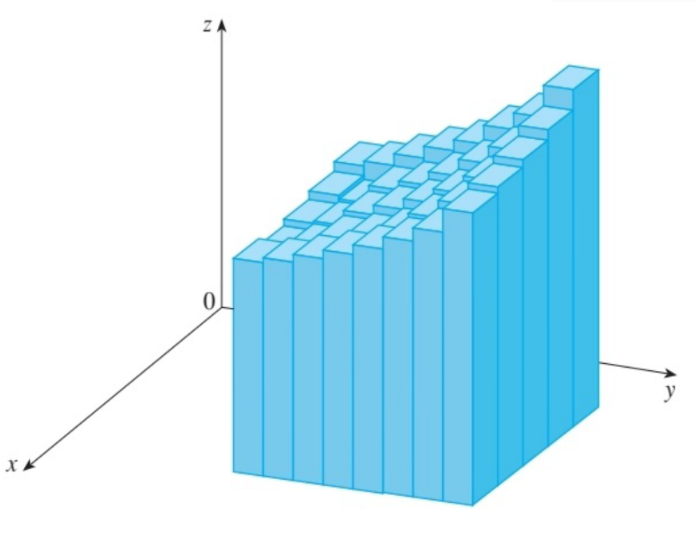
It can be computed as a limit

$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} f(x_i^*, y_j^*)\, \Delta A$$

where R is divided into $m \times n$ rectangles, (x_i^*, y_j^*) is a point chosen in the ijth rectangle, $f(x_i^*, y_j^*)$ is the height of the rectangular prism over that rectangular base and ΔA is the area of each small rectangle:





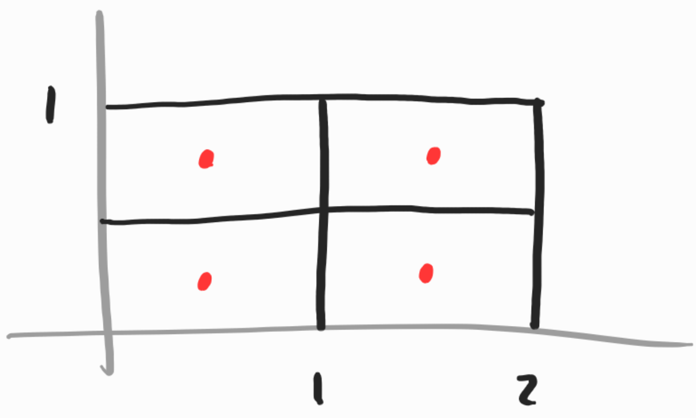


The definition above is a straightforward generalization of the single variable Riemann sum, but it can be much more involved to compute.

$\boxed{\text{Ex}}$ Let's approximate the volume

$$\iint_R x^2 y \, dA$$

where $R = [0,2] \times [0,1]$:



Let's use 4 rectangles $(m = n = 2)$ and the midpoints of these rectangles for (x_i^*, y_j^*):

$\left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{1}{4}\right), \left(\frac{3}{2}, \frac{3}{4}\right).$

Then $\Delta A = \Delta x \Delta y = \frac{1}{2}$ and

$$\sum_{i,j=1}^{2} f(x_i^*, y_j^*) \Delta A =$$

$$= \frac{1}{2}\left(f\left(\frac{1}{2}, \frac{1}{4}\right) + f\left(\frac{1}{2}, \frac{3}{4}\right) + f\left(\frac{3}{2}, \frac{1}{4}\right) + f\left(\frac{3}{2}, \frac{3}{4}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right) + \left(\frac{3}{2}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2\left(\frac{3}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{16} + \frac{3}{16} + \frac{9}{16} + \frac{27}{16}\right)$$

$$= \frac{40}{32} = \frac{5}{4}$$

By magic, the true volume is

$$\iint_R x^2 y \, dA = \frac{4}{3}.$$

Q: How can we compute a double integral without evaluating a limit?

Inspiration: the volume of a 3-dimensional region can be found by "scanning through" the region (think: MRI) and adding up all the cross sectional areas.

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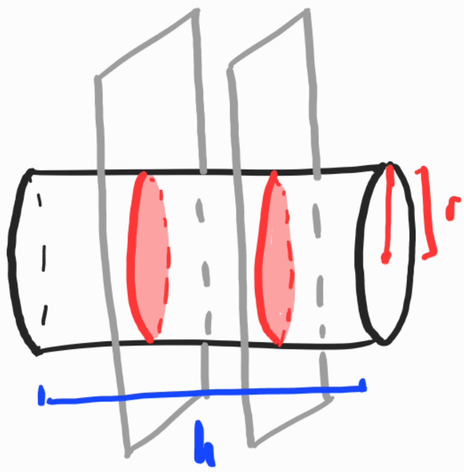
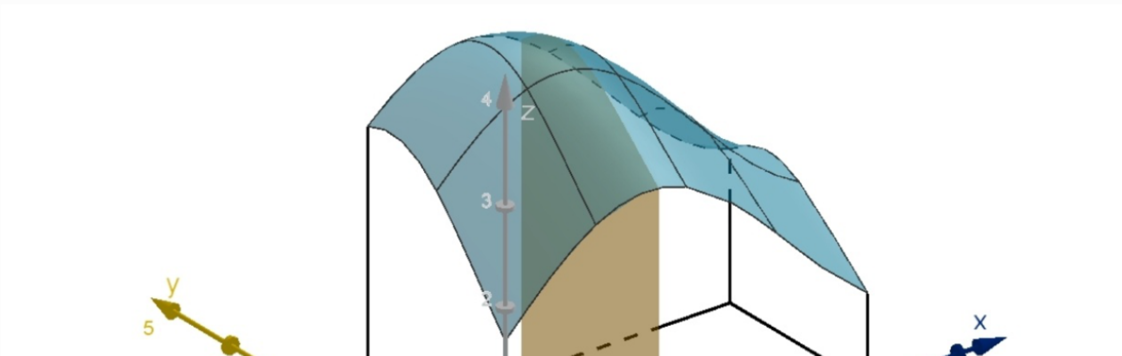
e.g. a cylinder with radius r and height h has circular cross sections



volume = $\pi r^2 h$

With a region under $z = f(x,y)$, we take cross sections:



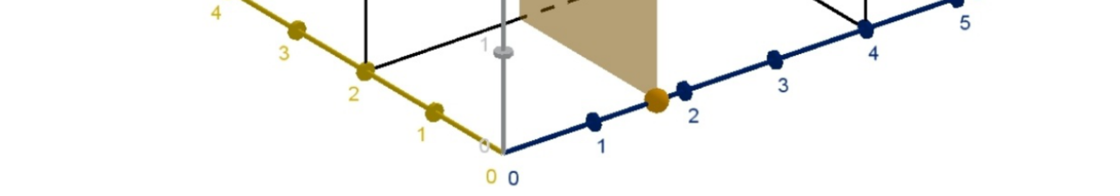


Figure: Cross sections of the region below the graph: $z = f(x, y)$

We can "add these up" with an

iterated integral:

$$\int_a^b A(x)\,dx$$

where for each $a \le x \le b$, $A(x)$ is

the area of the cross section at x:

$$A(x) = \int_c^d f(x,y)\,dy$$

x is constant, so

this is a single variable integral

Putting this together, we have:

$\boxed{\text{Theorem}}$ For a function $f(x,y)$ with a well-defined double integral over $R = [a,b] \times [c,d]$,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

$\boxed{\text{Ex}}$ For $f(x,y) = x^2 y$ on $R = [0,2] \times [0,1]$, the true volume is

$$\iint_R f(x,y)\,dA = \int_0^2 \underbrace{\int_0^1 x^2 y\,dy}_{\text{integrate } y}\,dx$$

$$= \int_0^2 \left[\frac{x^2 y^2}{2} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left[\frac{x^2}{2} - 0 \right] dx$$

now integrate x

$$= \frac{x^3}{6} \Big|_{x=0}^{2} = \frac{8}{6} - 0 = \frac{4}{3}.$$

Q: What happens if we take cross sections in the other direction?

$\boxed{\text{Theorem (Fubini)}}$ If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

$\boxed{\text{Ex}}$ Find the volume under the graph of $f(x,y) = ye^{xy}$ over

$$R = [0,1] \times [-1,2].$$

In iterated integral notation, this is

$$\iint_R f(x,y)\,dA = \int_0^1 \int_{-1}^2 ye^{xy}\,dy\,dx.$$

Unfortunately, $\int ye^{xy}\,dy$ requires integration

by parts, but we can use Fubini's

by parts, but we can use Fubini's

Theorem to switch the order of integration:

$$\int_0^1 \int_{-1}^2 ye^{xy}\,dy\,dx = \int_{-1}^2 \int_0^1 ye^{xy}\,dx\,dy$$

$$= \int_{-1}^2 \left[e^{xy}\right]_{x=0}^{x=1} dy$$

$$= \int_{-1}^2 \left[e^y - 1\right] dy$$

$$= e^y - y \Big|_{y=-1}^{y=2}$$

$$= (e^2 - 2) - (e^{-1} + 1)$$

$$= e^2 - e^{-1} - 3.$$

Exercise 1: For each of the following functions f and rectangles R, compute

$$\iint_R f(x,y)\,dA$$

using any method.

(a) $f(x,y) = 2x - 4y^3$, $R = [-5,4] \times [0,3]$

(b) $f(x,y) = x\sec^2(y)$, $R = [-2,3] \times [0, \frac{\pi}{4}]$

(c) $f(x,y) = \dfrac{1}{(2x+3y)^2}$, $R = [0,1] \times [1,2]$

One more helpful formula:

$\boxed{\text{Theorem}}$ If $f(x,y) = g(x)h(y)$ for two single variable functions g, h, then

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

if $R = [a,b] \times [c,d]$.

Next time: integrating over non-rectangular regions.