Lecture 15.2

Last time:

- $\iint_R f(x,y)\,dA$ represents the (signed) volume of the region under the graph of $f(x,y)$ over the rectangle $R = [a,b] \times [c,d]$ in $\mathbb{R}^2$.

- This can be computed with a double summation and a double limit, OR

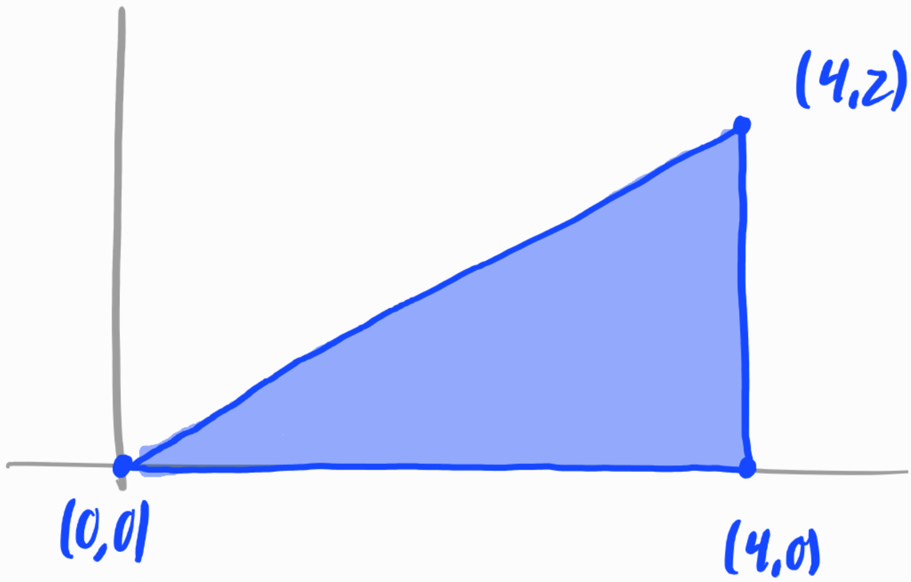
- Theorem: $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$.

- If f is continuous on R, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Q: What if we want to integrate over a region R that's not a rectangle?

Ex Let's find $\iint_R 4xy\,dA$ where R is the triangle with vertices $(0,0)$, $(4,0)$ and $(4,2)$.



Last time, we figured out that it's

useful to slice the

useful to slice the 3-dimensional region under $z = 4xy$ into cross sections and compute

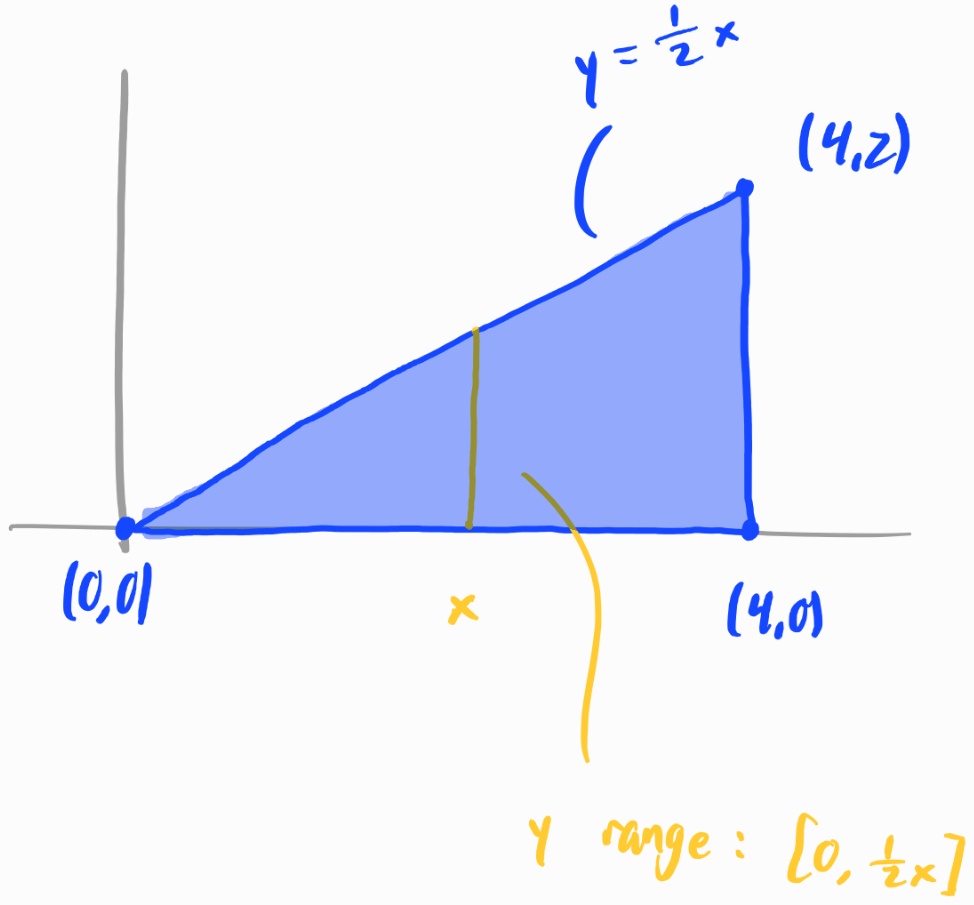
$$\iint_R 4xy \, dA = \int_0^4 A(x) \, dx$$

where $A(x)$ is the area of the cross section at x.

Not much changes here.

$$A(x) = \int_0^{??} 4xy \, dy$$

except the range of y-values for each cross section changes as we scan through the x-values.



The formula for $A(x)$ is

$$A(x) = \int_0^{\frac{1}{2}x} 4xy \, dy .$$

Let's put this into the iterated integral and see what happens:

$$\iint 4xy \, dA = \int^4 \int^{\frac{1}{2}x}$$

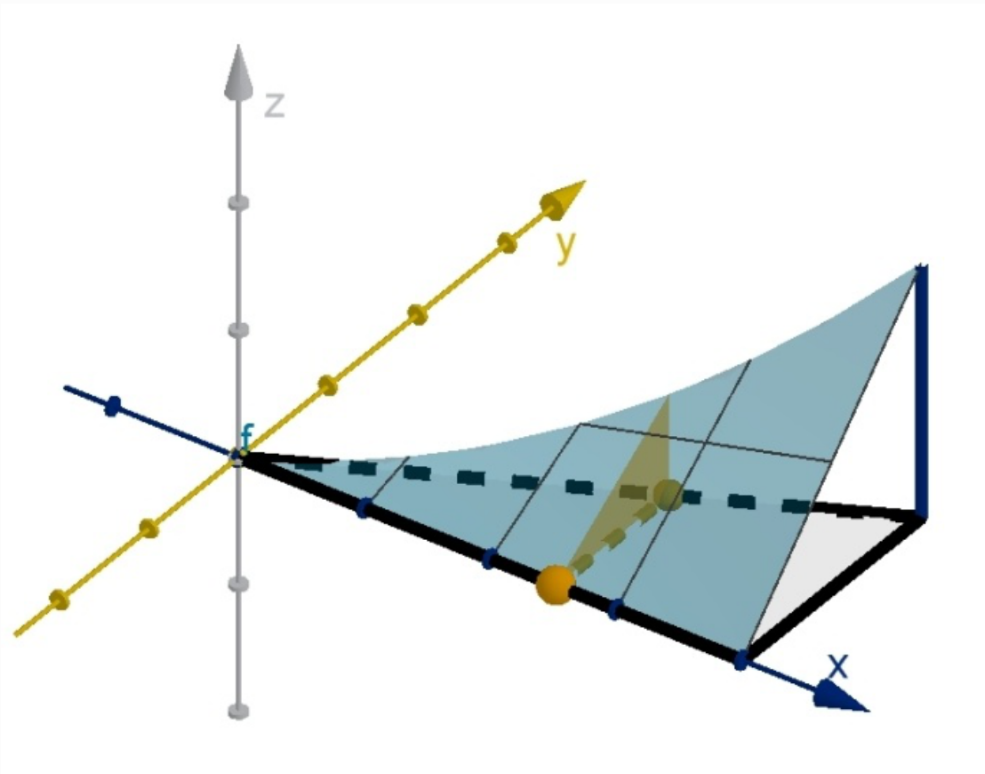
$$\iint_R 4xy\,dA = \int_0 \int_0 4xy\,dy\,dx$$

$$= \int_0^4 \left[2xy^2 \right]_{y=0}^{y=\frac{1}{2}x} dx$$

$$= \int_0^4 \frac{1}{2}x^3\,dx$$

we're left with a function of x only!

$$= \frac{1}{8}x^4 \Big|_{x=0}^{x=4} = 32.$$

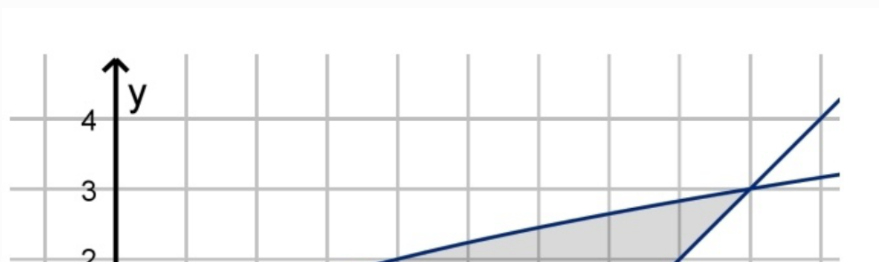


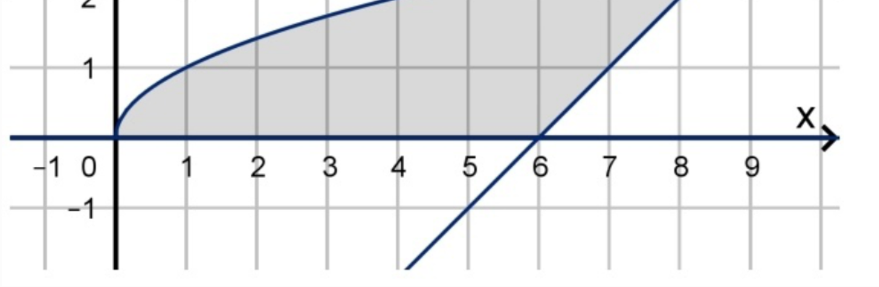
General strategy: describe the "upper" and "lower" boundaries of R as functions of x, then use these as bounds in the inner integral (with dy).

Ex Let R be the region in the xy-plane bounded by $y = \sqrt{x}$, $y = x - 6$ and the x-axis. Let's find

$$\iint_R (x+y)\, dA.$$

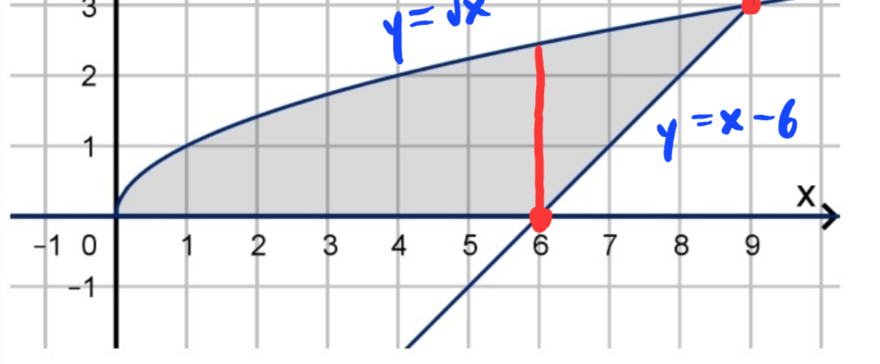
Here's a plot of the region:





Notice that there isn't a single "bottom boundary" this time. For x between 0 and the x-intercept of $y = x-6$, the top function is $y=\sqrt{x}$ and the bottom is $y=0$, but for the rest of the x-values, $y=\sqrt{x}$ is on top and $y = x-6$ is on bottom.

Let's break up the region like this:



It appears that the x-intervals are $[0,6]$ and $[6,9]$, but let's verify:

- $x-6=0$ when $x=6$ ✓
- $\sqrt{x}=x-6 \Rightarrow x-\sqrt{x}-6=0$

$$\Rightarrow (\sqrt{x}-3)(\sqrt{x}+2)=0$$

$$\Rightarrow \sqrt{x}=3 \Rightarrow x=9 \quad ✓$$

So our double integral splits up as:

$$\iint_R (x+y)\,dA = \int_0^6\int_0^{\sqrt{x}} (x+y)\,dy\,dx + \int_6^9\int_{6-x}^{\sqrt{x}} (x+y)\,dy\,dx$$

$$= \int_0^6 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx + \int_6^9 \left[xy + \frac{y^2}{2} \right]_{y=6-x}^{y=\sqrt{x}} dx$$

$$= \int_0^6 \left[x^{3/2} + \frac{x}{2} \right] dx + \int_6^9 \left[x^{3/2} + \frac{x}{2} - x(6-x) - \frac{(6-x)^2}{2} \right) dx$$

$$= \left[\frac{x^{5/2}}{5/2} + \frac{x^2}{4} \right]_0^6 + \int_6^9 \left[x^{3/2} + \frac{x}{2} + \frac{x^2}{2} - 18 \right) dx$$

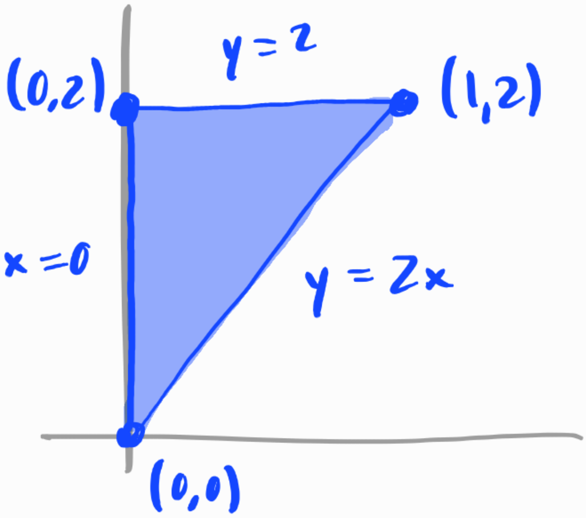
$$= \frac{2 \cdot 6^{5/2}}{5} + 9 + \left[\frac{x^{5/2}}{5/2} + \frac{x^2}{4} + \frac{x^3}{6} - 18x \right]_6^9$$

$$= \frac{2 \cdot 6^{5/2}}{5} + 9 + \left(\frac{2 \cdot 3^5}{5} + \frac{81}{4} + \frac{9^3}{6} - 162 \right)$$

$$- \left(\frac{2 \cdot 6^{5/2}}{5} + 9 + 36 - 108 \right)$$

$$= \frac{2979}{20} = 148.95.$$

$\boxed{\text{Ex}}$ Compute $\iint_R e^{y^2} \, dA$ where R is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,2)$.



There's only one top and bottom function, so:

$$\iint_R e^{y^2}\, dA = \int_0^1 \int_{2x}^2 e^{y^2}\, dy\, dx.$$

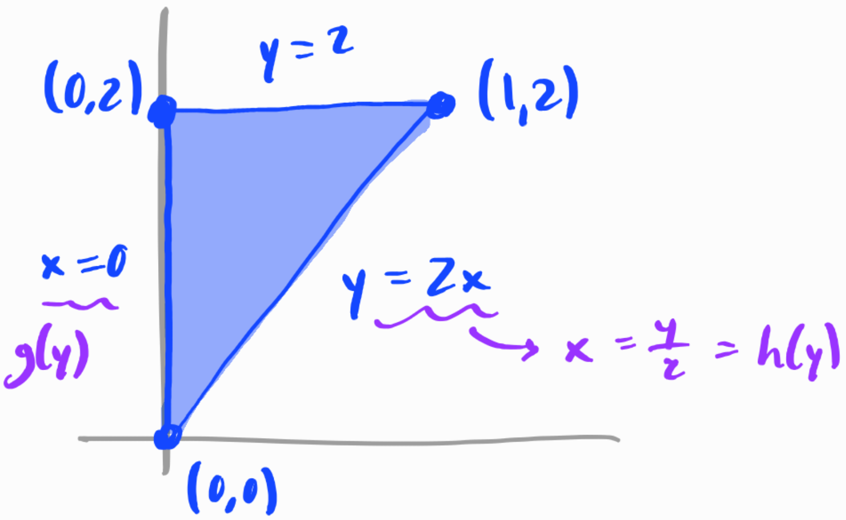
However, e^{y^2} doesn't have a known antiderivative, so we need to switch the order.

WARNING: We cannot just switch the x- and y-bounds unless R is a rectangle!

Instead, we need to write

$$\int_a^b \int_{g(y)}^{h(y)} e^{y^2}\, dx\, dy$$

where $a \le y \le b$ and $g(y) \le x \le h(y)$.



This way our integral can be written

$$\int_0^2 \int_0^{y/2} e^{y^2}\, dx\, dy = \int_0^2 \left[x e^{y^2} \right]_{x=0}^{x=y/2} dy$$

$$= \int_0^2 \left[\frac{y}{2} e^{y^2} \right] dy$$

$$= \left[\frac{1}{4} e^{y^2}\right]_0^2$$

$$= \frac{e^4}{4} - \frac{1}{4} = \frac{e^4 - 1}{4} \approx 13.4.$$

Exercise 1: Compute $\iint_R f(x,y)\, dA$ where

(a) $f(x,y) = xy - x^2$, R is bounded by $y = x^2$ and $y = 2x$

(b) $f(x,y) = x^2 + y^2$, R is the square with corners $(\pm 1, \pm 1)$

(c) $f(x,y) = e^{x+y}$, R is bounded by $x = -1$, $x = 1$, $y = x$ and $y = 2x$

(d) $f(x,y) = 2 + \cos(y^2)$, R is the ...

R is the triangle with vertices $(0,0)$, $(0,2)$, $(6,2)$

Applications

Proposition The area of a region R in the xy-plane is equal to

$$\iint_R 1 \, dA.$$

Pf:

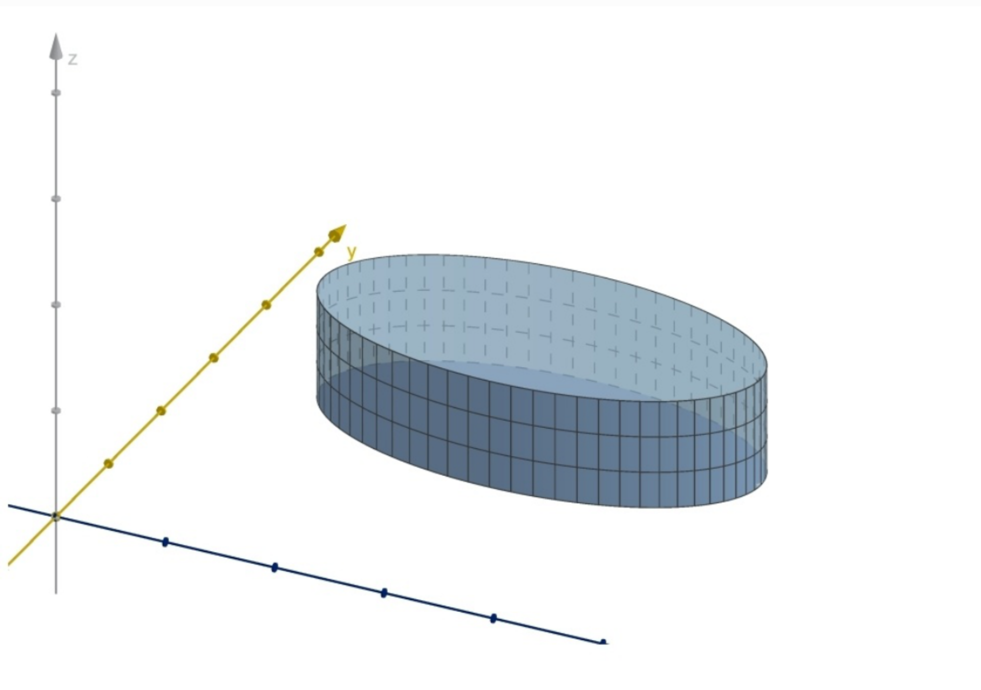


Figure: A solid of height 1 over a domain D $\square$

Def: The average value of $f(x,y)$ over a region R is

$$\frac{1}{A(R)} \iint_R f(x,y)\, dA$$

where $A(R)$ is the area of R.

Exercise 2: Find the area of the region between $y = 1 - x^2$ and $y = x^2 - 3$ using a double integral.

Exercise 3: Find the average value of

$f(x,y) = x^2$ over the region in Exercise 2.

Next time: polar coordinates.