Lecture 15.2

Last time:

- The L-function of a character $\chi \in X(b)$ is the Dirichlet series

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

- By expressing $f_{a,b}$ as a linear combination of characters mod b,

$$f_{a,b} = \sum_{\chi \in X(b)} \frac{\chi(a)^{-1}}{\phi(b)} \chi$$

we can situate the sum $\sum_{p \equiv a \pmod{b}} \frac{1}{p^s}$ among pieces of these L-functions:

$$\sum_{p \equiv a \pmod{b}} \frac{1}{p^s} = \frac{1}{\phi(b)} \sum_{\chi} \chi(a)^{-1} H(\chi, s)$$

where

$$\log\left(\prod_{\chi} L(\chi, s)\right) = \sum_{\chi} H(\chi, s) + G(s).$$

- Our goals then are to show that $G(s)$ is bounded as $s \to 1^+$ while $\prod_{\chi} L(\chi, s)$ diverges as $s \to 1^+$.

Let's start with $G(s)$:

$$G(s) = \sum_{k=2}^{\infty} \sum_{\chi} \sum_{p} \frac{\chi(p^k)}{k} p^{-sk}$$

$$= \sum_{\chi} G(\chi, s)$$

where $G(\chi, s) = \sum_{k=2}^{\infty} \sum_{p} \frac{\chi(p^k)}{k} p^{-sk}$.

In particular, $\log L(\chi, s) = H(\chi, s) + G(\chi, s)$.

Notice that

$$|G(\chi, 1)| = \left| \sum_{k=2}^{\infty} \sum_{p} \frac{\chi(p^k)}{kp^k} \right|$$

$$\leq \sum_{k=2}^{\infty} \sum_{p} \frac{1}{kp^k} \quad \text{since } |\chi(p^k)| = 1$$

$$\leq \sum_{k=2}^{\infty} \sum_{p} \frac{1}{p^k} = \sum_{p} \sum_{k=2}^{\infty} \frac{1}{p^k}$$

$$= \sum_{p} \frac{1/p}{1-p} = \sum_{p} \frac{1}{p^2} \left(\frac{p}{1-p} \right)$$

$$\leq \sum_{p} \frac{2}{p^2} \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = 2\zeta(2) = \frac{\pi^2}{3}.$$

So each $G(\chi,1)$ converges absolutely and hence

so does $G(1) = \sum_{\chi} G(\chi,1)$.

Next, let's deal with the $H(\chi,s)$ terms.

Once again,

$$\sum_{p \equiv a \pmod b} \frac{1}{p^s} = \frac{1}{\phi(b)} \sum_{\chi} \chi(a)^{-1} H(\chi,s) + G(s)$$

$$= \frac{1}{\phi(b)} \cancelto{1}{\chi_0(a)^{-1}} H(\chi_0,s) + \sum_{\chi \neq \chi_0} \frac{\chi(a)^{-1}}{\phi(b)} H(\chi,s) + G(s)$$

where χ_0 is the trivial character mod b.

But $H(\chi_0,s) = \sum_{p} \chi_0(p)\, p^{-s}$

$$= \sum \quad p^{-s} \qquad \sum \frac{1}{}$$

$$\sum_{(p,b)=1} \frac{1}{p^s} = \sum_{p \nmid b} \frac{1}{p^s}$$

and since there are infinitely many primes $p \nmid b$, $H(\chi_0, s)$ diverges as $s \to 1$.

So it all comes down to this: can we show that every other $H(\chi, s)$ converges as $s \to 1^+$?

If so, we'd be done.

Here is a summary of the things we need to check.

$\boxed{\text{Theorem}}$ Let $b \geq 2$ and let χ be a nontrivial character mod b. Then

(1) $L(\chi,s)$ converges for all $\mathrm{Re}(s) > 0$.

In particular, $L(\chi,1)$ exists.

(2) $L(\chi,1) \neq 0$, so $\log L(\chi,1)$ exists.

(3) $H(\chi,s)$ is bounded as $s \to 1^+$.

Pf: (1) I'll skip this; let me know if you want the details.

(2) Suppose $L(\chi,1) = 0$ for some $\chi \neq \chi_0$.

Looking in the factorization

$$\varphi_b(s) = \prod_{\chi} L(\chi,s)$$

this would imply $\zeta_b(1)$ exists, since the 0 from $L(\chi, 1)$ would cancel out the divergent term $L(\chi_0, 1)$. — We haven't show this, but $L(\chi_0, s) - \frac{1}{s-1}$ converges for $s = 1$, which means there's only one "infinity" to cancel out at $s = 1$. This can be made more precise using complex analysis.

Let's show this is not the case.

We know $\zeta_b(s)$ has a product formula:

$$\zeta_b(s) = \prod_{\chi} \prod_{p} \frac{1}{1 - \chi(p) p^{-s}}$$

$$= \prod \prod \Big(\quad \Big)^{-1}$$

$$\prod_{p \nmid b} \prod_{\chi} \left(1-\chi(p)p^{-s}\right).$$

$\chi(p) = 0$ if $p \mid b$

We also know each $\chi(p)$ is a root of unity

ω, $\omega^k = 1$, for some $k \mid \phi(b)$.

For each $p \nmid b$, let v_p be the order of

p mod b, i.e. $p^{v_p} \equiv 1 \pmod{b}$ but

$p^k \not\equiv 1 \pmod{b}$ for $k < v_p$.

Then $1 - z^{v_p} = \prod_{\omega^{v_p}=1} (1-\omega z)$ and for

each v_pth root of unity ω, there are

exactly $\frac{\phi(b)}{v_p}$ characters $\chi \in X(b)$ with

$\chi(p) = \omega$.

That is,

$$\prod_{\chi} (1-\chi(p)z) = (1-z^{v_p})^{\frac{\phi(b)}{v_p}}$$

which implies

$$\zeta_b(s) = \prod_{p\nmid b} \prod_{\chi} (1-\chi(p)p^{-s})^{-1}$$

$$= \prod_{p\nmid b} (1-p^{-sv_p})^{-\frac{\phi(b)}{v_p}}.$$

Notice that each $(1-p^{-sv_p})^{-\frac{\phi(b)}{v_p}}$ is a power of a geometric series in $x = p^{-sv_p}$, hence a Dirichlet series with <u>nonnegative coefficients</u>.

Any such Dirichlet series can be extended to a convergent function for $Re(s) > 0$.

However,

$$\varphi_b(s) = \prod_{p \nmid b} \left(\frac{1}{1 - p^{-s v_p}} \right)^{\frac{\phi(b)}{v_p}}$$

$$\geq \prod_{p \nmid b} \frac{1}{1 - p^{-s\phi(b)}}$$

$$= \sum_{(n,b)=1} \frac{1}{n^{\phi(b)s}}$$

which implies for $s = \frac{1}{\phi(b)} < 1$,

$$\varphi_b\left(\frac{1}{\phi(b)}\right) \geq \sum_{(n,b)=1} \frac{1}{n} = \infty .$$

This is a contradiction, so $L(\chi, s) \neq 0$ for <u>any</u> $\chi \neq \chi_0$.

(3) We want to show that each

$$H(\chi, s) = \sum_p \chi(p) p^{-s}$$

is bounded as $s \to 1^+$, as long as $\chi \neq \chi_0$.

However, recall that

$$\log L(\chi, s) = H(\chi, s) + G(\chi, s).$$

Since $\log L(\chi, 1)$ exists by (2) and $G(\chi, s)$ is bounded as $s \to 1^+$ by previous work, $H(\chi, s)$ must also be bounded as $s \to 1^+$. $\square$

Putting it all together:

Theorem (Dirichlet) For any $b \geq 2$ and $a \in \mathbb{N}$ with $\gcd(a,b) = 1$, there are infinitely many primes $p \equiv a \pmod{b}$.

Pf: We know that for any $\mathrm{Re}(s) > 1$,

$$\sum_{p \equiv a \,(\mathrm{mod}\, b)} \frac{1}{p^s} = \frac{1}{\phi(b)} \sum_{p \nmid b} \frac{1}{p^s} + \sum_{\chi \neq \chi_0} \frac{\chi(a)^{-1}}{\phi(b)} H(\chi, s) + G(s).$$

But $\sum_{p \nmid b} \frac{1}{p}$ diverges while $G(s)$ and each $H(\chi, s)$ are bounded as $s \to 1^+$.

This means $\sum_{p \equiv a \,(\mathrm{mod}\, b)} \frac{1}{p}$ diverges, so there are infinitely many terms in the sum. $\square$

Next time: applications of L-functions.