Lecture 15.3

Last time:

- To integrate over $R = [a,b] \times [c,d]$, use an iterated integral

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

- To integrate over a non-rectangular region R, write down a top and a bottom function $y = g(x)$ and $y = h(x)$ and compute

$$\iint_R f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx.$$

- If this doesn't work, you may need to

- break up the region into smaller pieces

- switch dy and dx and find right and left functions $x = g(y)$, $x = h(y)$ to compute

$$\int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dx\,dy.$$

Polar Coordinates

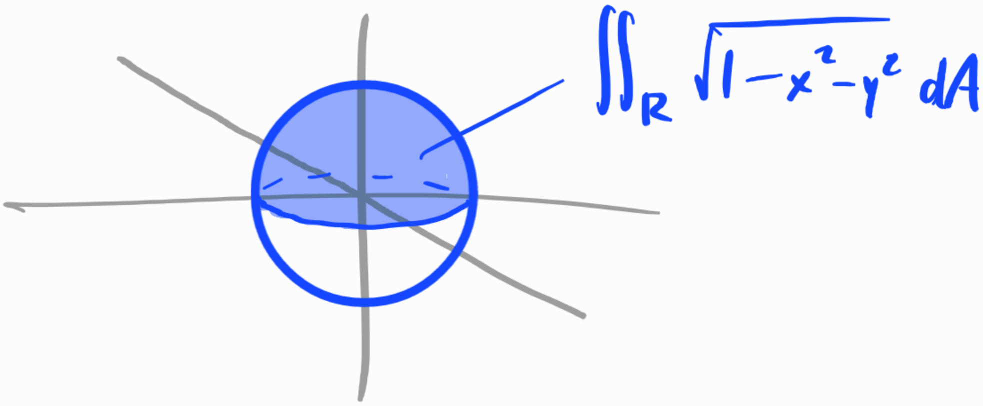
Motivation: let's find the volume of the unit sphere $x^2+y^2+z^2=1$.

Solving for z, we get two functions:

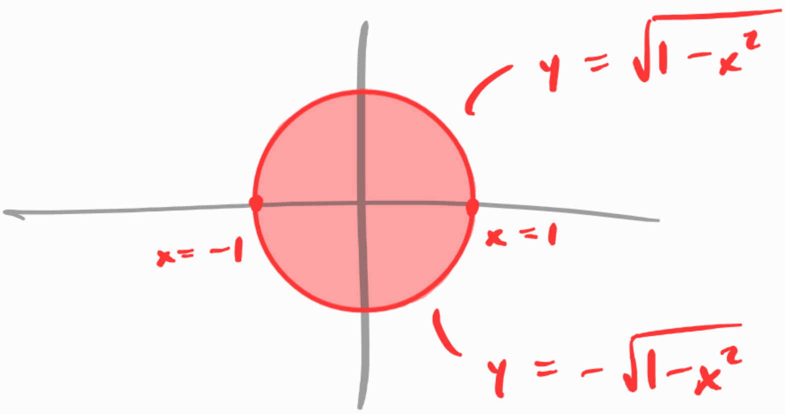
$$z = \sqrt{1-x^2-y^2} = f(x,y)$$

$$z = -\sqrt{1-x^2-y^2}$$

but the volume of the full sphere is twice the volume just under $f(x,y)$ alone.



The region R we need to integrate over is the unit circle in $\mathbb{R}^2$:



So our volume double integral is:

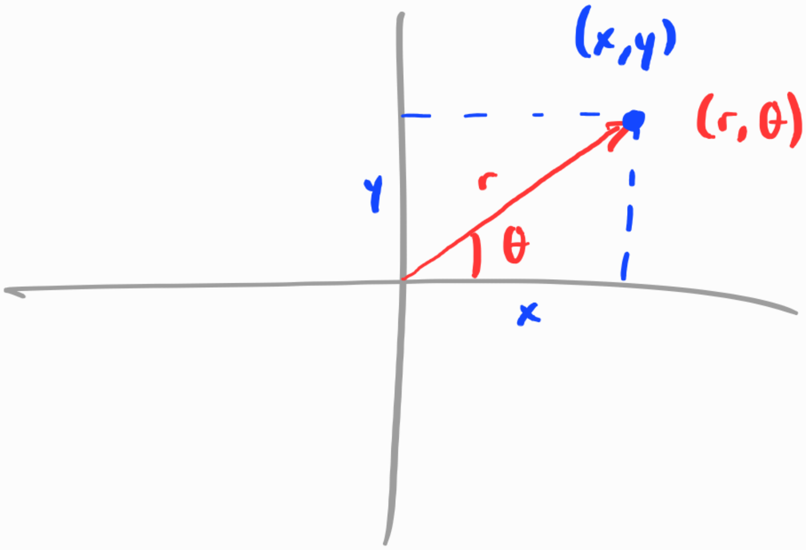
$$\text{volume} = 2 \iint_R \sqrt{1 - x^2 - y^2}\, dA$$

$$= 2 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx.$$

Exercise 1: Try to use trig substitution to solve this integral.

This isn't particularly nice, so let's find another way.

Key insight: points in $\mathbb{R}^2$ are represented by (x, y) or alternatively by a unique radius $r \geq 0$ and angle θ with the positive x-axis:



Here's how to get back and forth:

$$(x,y) \rightsquigarrow \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

$$(r,\theta) \rightsquigarrow \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

These new coordinates (r,θ) form the polar coordinate system.

Exercise 2: Convert these points to polar coordinates:

(a) $(1, 0)$ (b) $(0, -5)$ (c) $(4, 3)$

(d) $(1, 1)$ (e) $(-2, 2)$ (f) $(1, -\sqrt{3})$

Exercise 3: Convert these polar coordinates back to standard xy-coordinates:

(a) $(1, \frac{\pi}{4})$ (b) $(5, 0)$ (c) $(2, \pi)$

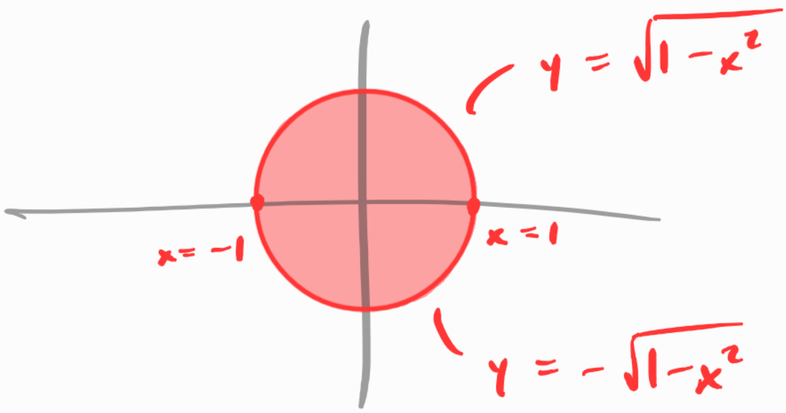
(d) $(\frac{1}{2}, \frac{\pi}{6})$ (e) $(4, \frac{2\pi}{3})$ (f) $(1, \frac{5\pi}{4})$

We can also convert functions back and forth between the coordinate systems, using

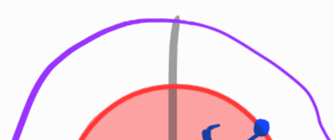
$$r = \sqrt{x^2+y^2} \qquad\qquad x = r\cos\theta$$

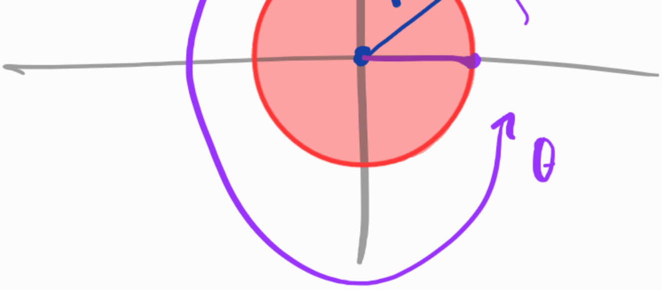
$$\theta = \arctan\left(\frac{y}{x}\right) \quad \longleftrightarrow \quad y = r\sin\theta$$

For example, our unit circle picture



can be written in polar coordinates as





$r = 0$ to $r = 1$ $\qquad$ $\theta = 0$ to $\theta = 2\pi$

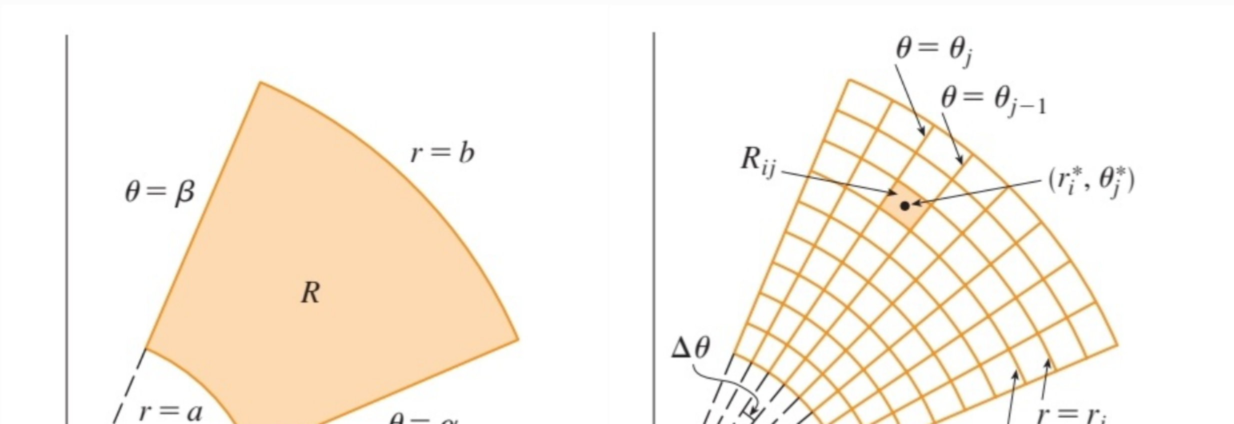
Also, since $r^2 = x^2 + y^2$ we can write

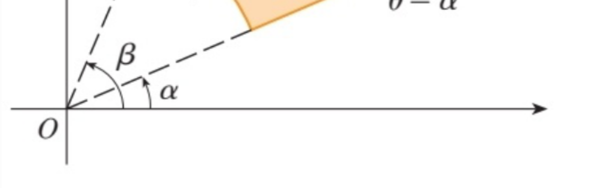
$f(x,y) = \sqrt{1 - x^2 - y^2}$ as a function of

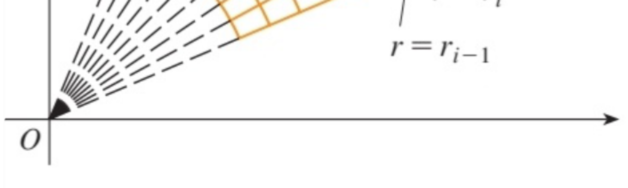
r and θ:

$$f(r,\theta) = \sqrt{1 - r^2}$$

Last step: how do we handle $dA = dy\,dx$?





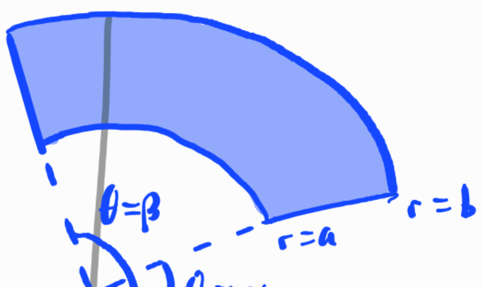


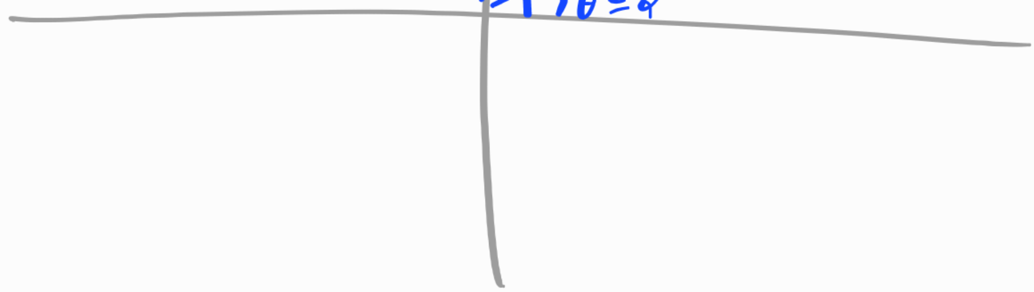
Idea: a "rectangle" in (r, θ) space is a section of a circle in (x,y) space and the transformation $(x,y) \rightsquigarrow (r,\theta)$ changes the differential by: $dA = dy\,dx \rightsquigarrow \underline{\underline{r}}\,dr\,d\theta$.

$\boxed{\text{Theorem}}$ If $f(x,y)$ is continuous on the polar region $R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta,\ 0 \le \beta - \alpha \le 2\pi \end{array} \right\}$

then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$





$\boxed{\text{Ex}}$ Let's finish our computation of the volume of the sphere.

We have $\iint_R \sqrt{1-x^2-y^2}\, dA$ where

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

and $f(r,\theta) = \sqrt{1-r^2}$. Then

$$\text{volume} = 2 \int_0^{2\pi} \int_0^1 r\sqrt{1-r^2}\, dr\, d\theta$$

$$= 2 \int_0^{2\pi} \left[\frac{-1}{3} (1-r^2)^{3/2} \right]_{r=0}^{r=1} d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{-1}{3}(1-1)^{3/2} + \frac{1}{3}(1-0)^{3/2}\right] d\theta$$

$$= 2\int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{4\pi}{3}.$$

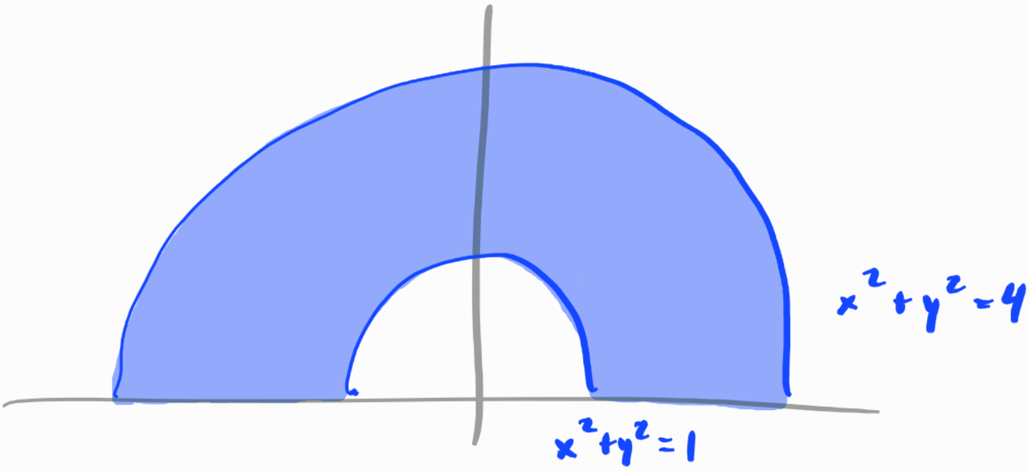
So the volume of the unit sphere is $\frac{4\pi}{3}$.

Exercise 4: Modify the above formula to calculate the volume of a sphere with radius $R > 0$.

Ex Let's use polar coordinates to find

$$\iint_R (3x + 4y^2)\, dA$$

where R is the region pictured below:



This R can be described by

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 1 \le r \le 2, \\ 0 \le \theta \le \pi \end{array} \right\}$$

so

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r \, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$

$$= \int_0^\pi \Big[r^3\cos\theta + r^4\sin^2\theta \Big]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \left[8\cos\theta + 16 \sin^2\theta - \cos\theta - \sin^2\theta \right] d\theta$$

$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

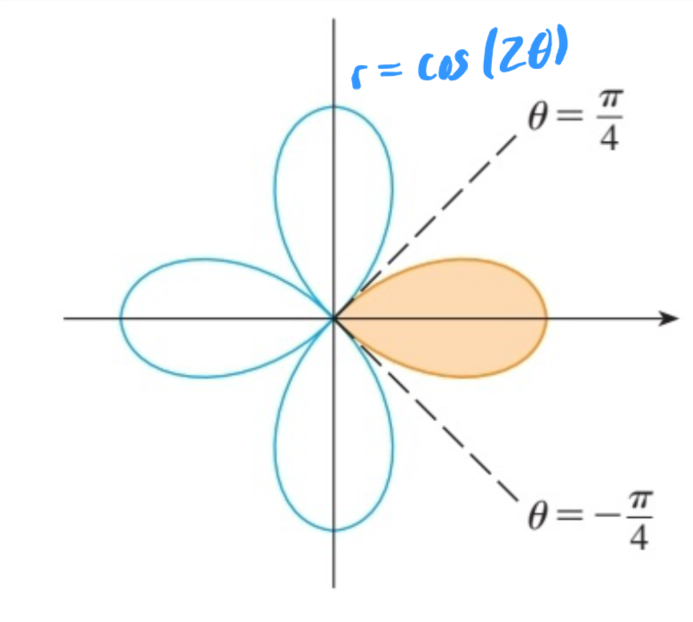
$$= \int_0^{\pi} \left[7\cos\theta + \frac{15}{2}(1 - \cos(2\theta)) \right] d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin(2\theta) \right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{15\pi}{2}.$$

What if R isn't a "polar rectangle", i.e. with r, θ bounded by constant values the whole time?

Ex The function $r = \cos(2\theta)$ traces a four-petal flower pattern in $\mathbb{R}^2$:



Let's find the total area of the figure by computing the area of one of its petals.

First, recall from Lecture 15.2 that

$$\text{area}(R) = \iint_R 1 \, dA.$$

Here, R is the region enclosed by one petal:

$$\{(r, \theta) : \ldots\ 0 \le r \le \cos(2\theta)\}$$

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 0 \le r \le \cos(2\theta) \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{array} \right\}$$

so

$$\text{area}(R) = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} \underline{\underline{r}} \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=\cos(2\theta)} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} \cos^2(2\theta) \right] d\theta$$

$$\cos^2(u) = \frac{1}{2}(1+\cos(2u))$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(1+\cos(4\theta)) \, d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{1}{16}\sin(4\theta) \right]_{\theta=-\pi/4}^{\theta=\pi/4}$$

$$= \frac{\pi}{8}.$$

So the total area enclosed by the graph

of $r = \cos(2\theta)$ is $\frac{\pi}{2}$.

Exercise 5: Try this example again with $r = \cos(n\theta)$ for different n.

Exercise 6: Compute each area or volume by changing to polar coordinates.

(a) The area inside the graph of $r = 3 + 2\sin\theta$ and outside the circle centered at the origin with radius 2.

(b) The volume inside the hemisphere

$$\{(x,y,z) \mid z \geq 0,\ x^2 + y^2 + z^2 \leq 9\}$$

and inside the cylinder $x^2+y^2=5$.

(c) The volume of the region bounded by the surface $z=x^2+y^2$ and the plane $z=16$.

Next time: applications of double integrals, plus triple integrals.