Lecture 15.4

Last time:

- We can convert (x,y) coordinates to (r,θ) polar coordinates and vice versa:

$$(x,y) \rightsquigarrow \left(\sqrt{x^2+y^2},\ \arctan\left(\frac{y}{x}\right)\right)$$

$$(r\cos\theta,\ r\sin\theta) \leftsquigarrow (r,\theta)$$

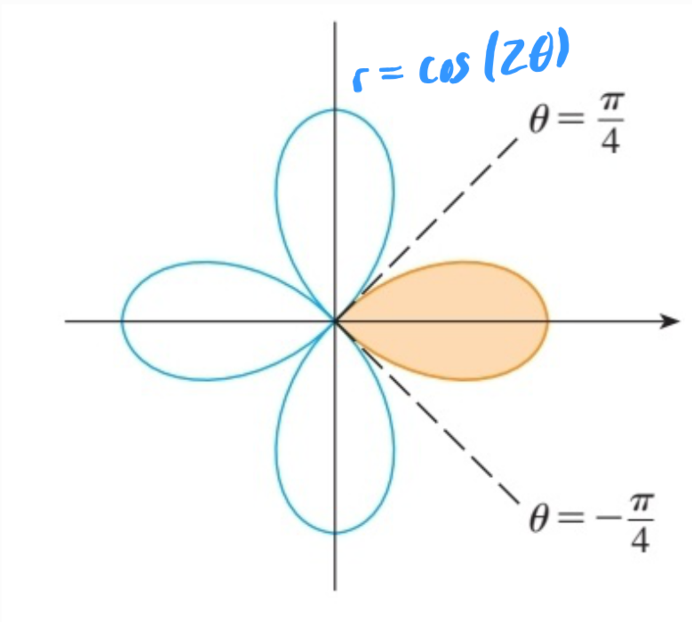
- To integrate a continuous $f(x,y)$ over a "polar rectangle"

$$R = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array} \right\},$$

use $dA = r\,dr\,d\theta$ and

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r,\theta)\,r\,dr\,d\theta.$$

$\boxed{\text{Ex}}$ The function $r = \cos(2\theta)$ traces a four-petal flower pattern in $\mathbb{R}^2$:



Let's find the total area of the figure

by computing the area of one of its petals.

First, recall from Lecture 15.2 that

$$\text{area}(R) = \iint_R 1 \, dA.$$

Here, R is the region enclosed by one petal:

$$R = \left\{ (r, \theta) \;\middle|\; \begin{array}{l} 0 \le r \le \cos(2\theta) \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{array} \right\}$$

so

$$\text{area}(R) = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} \underline{\underline{r}} \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=\cos(2\theta)} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} \cos^2(2\theta) \right] d\theta$$

$\cos^2(u) = \frac{1}{2}(1 + \cos(2u))$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(1 + \cos(4\theta))\, d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{1}{16}\sin(4\theta)\right]_{\theta = -\pi/4}^{\theta = \pi/4}$$

$$= \frac{\pi}{8}.$$

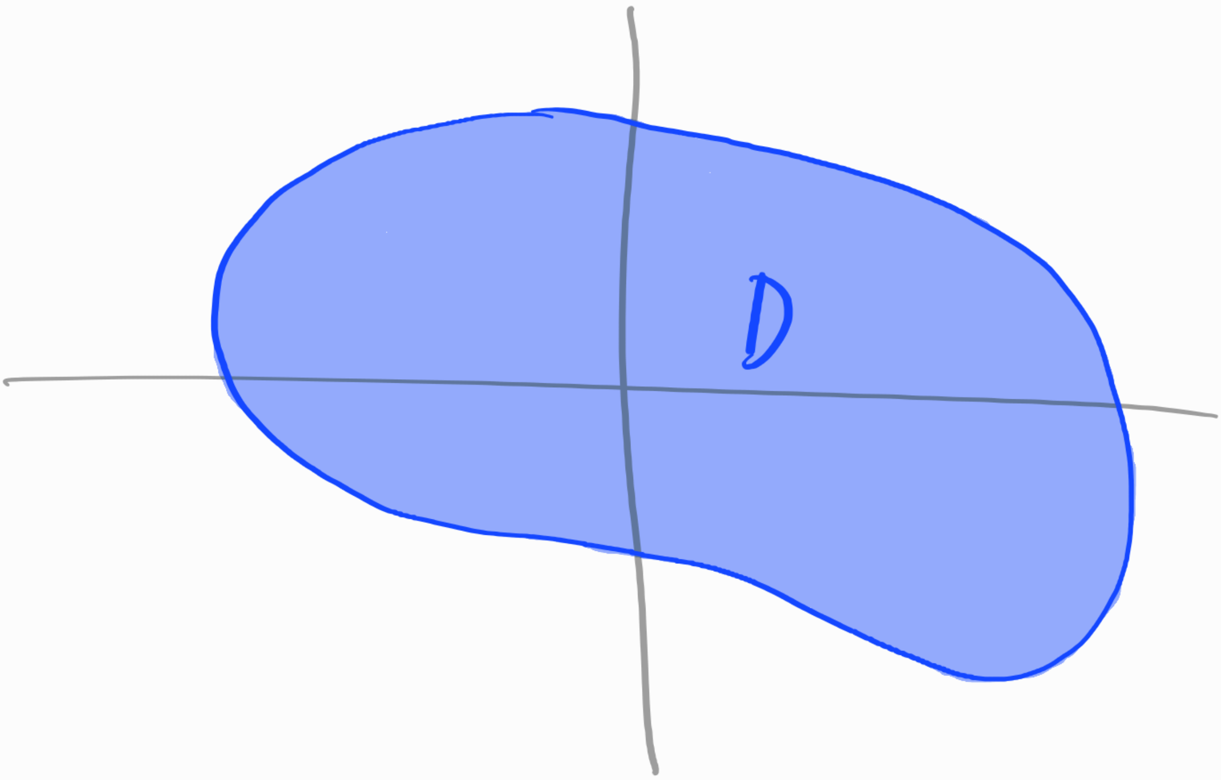
So the total area enclosed by the graph of $r = \cos(2\theta)$ is $\frac{\pi}{2}$.

Exercise 1: Try this example again with $r = \cos(n\theta)$ for different n.

Applications of Double Integrals

Mass

Let D be a thin plate made of some material whose density varies across D.



If $\rho(x,y)$ is the density of D at (x,y), then the total mass of D is

$$m(D) = \iint \rho(x,y)\,dA$$ density

$$M(D) = \iint_D \rho(x,y)\, dA \quad \approx \quad \text{density} \times \text{area}$$

$$\text{or} \quad \text{density} = \frac{\text{mass}}{\text{area}}$$

Exercise 2: Find the mass of

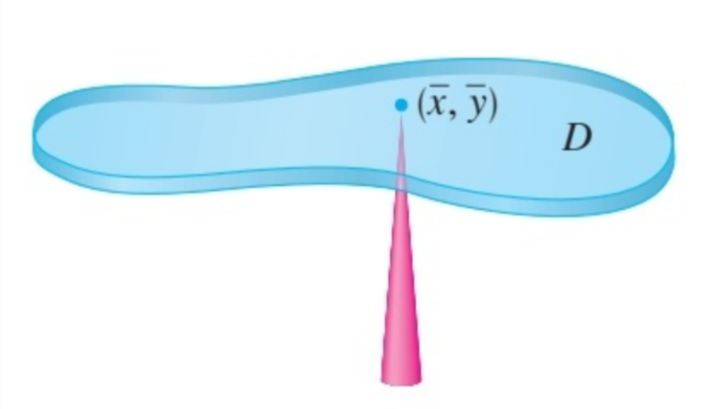
$$D = \{(x,y) \mid x^2+y^2 \leq 2,\ x,y \geq 0\}$$

if its density at (x,y) is given by

$$\rho(x,y) = e^{x^2+y^2}.$$

Center of mass

The center of mass of D is, loosely, the point $(\bar{x}, \bar{y})$ in D where D would balance perfectly if it were placed on a needle at that point.



If D has density function $\rho(x,y)$, the coordinates of its center of mass are given by

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x,y)\, dA$$

and $$\bar{y} = \frac{1}{m} \iint_D y \rho(x,y)\, dA$$

where m is the mass of D.

Exercise 3: Find the center of mass of the triangle D with vertices (0,0), (1,0) and (0,2) and density function

$$\rho(x,y) = x + y^2 + 1.$$

Probability

We can interpret probabilities in the same framework as mass and density.

Suppose D is a region of possible outcomes (x,y) to two random variables X and Y.

If $f(x,y)$ is the probability function for D, then D should have "total mass" 1:

$$P((X,Y) \text{ in } D) = \iint_D f(x,y)\, dA = 1$$

and for any subset $R \subseteq D$, the probability of choosing a point at random in R is the "total mass of R":

$$P((X,Y) \text{ in } R) = \iint_R f(x,y)\, dA.$$

For example,

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

The expected values of X and Y are

just the coordinates of the "center of mass" of D:

$$E[X] = \iint_D x f(x,y)\, dA$$

$$E[Y] = \iint_D y f(x,y)\, dA.$$

Exercise 4: For each D and $f(x,y)$, check that f is a probability function on D, then compute the expected values of X and Y.

(a) $D = [0,1] \times [0,1]$, $f(x,y) = x + \frac{3}{2}y^2$

(b) $D = \{(x,y) \mid 0 \leq y \leq x \leq 1\}$, $f(x,y) = 10x^2y$

(c) $D = \{(x,y) \mid x, y \geq 0\}$, $f(x,y) = 6e^{-2x-3y}$

(Hint: D is an infinite region, so you will need to use limits to evaluate the resulting improper integrals.)

(d) $D = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$, $f = 6xy$

Triple Integrals

Motivation: what if we want compute the mass of a 3-dimensional region, or the joint probability of 3 random variables?

Def Let $f(x,y,z)$ be a function on a region

R in $\mathbb{R}^3$. The triple integral of f over R is the mass of R if its density at (x,y,z) is $f(x,y,z)$, written

$$\iiint_R f(x,y,z)\,dV.$$

Two ways to calculate:

- Triple Riemann sum:

$$\iiint_R f(x,y,z)\,dV = \lim_{l,m,n \to \infty} \sum_{\substack{1 \le i \le l \\ 1 \le j \le m \\ 1 \le k \le n}} f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

where $\Delta V = \Delta x \Delta y \Delta z$ is the volume of a small rectangular prism subdividing R.

• Iterated integral: if $R = [a,b] \times [c,d] \times [e,f]$,

$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx.$$

Fubini's Theorem for triple integrals says you can switch dx, dy and dz, as long as R is a rectangular prism and f is continuous.

$\boxed{\text{Ex}}$ Let's compute $\iiint_R xyz^2\,dV$ where

$$R = [0,1] \times [-1,2] \times [0,3].$$

We can integrate in any order, so let's

choose the "usual" one:

$$\iiint_R xyz^2\,dV = \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dy\,dx$$

$$= \int_0^1\int_{-1}^2 \left[\frac{xyz^3}{3}\right]_{z=0}^{z=3} dy\,dx$$

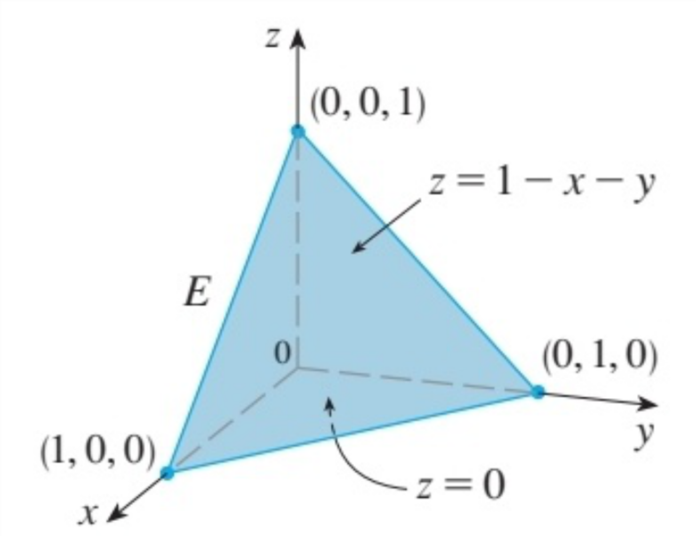
$$= \int_0^1\int_{-1}^2 9xy\,dy\,dx$$

$$= \int_0^1 \left[\frac{9xy^2}{2}\right]_{y=-1}^{y=2} dx$$

$$= \int_0^1 \left[18x - \frac{9}{2}x\right]dx = \int_0^1 \frac{27x}{2}\,dx$$

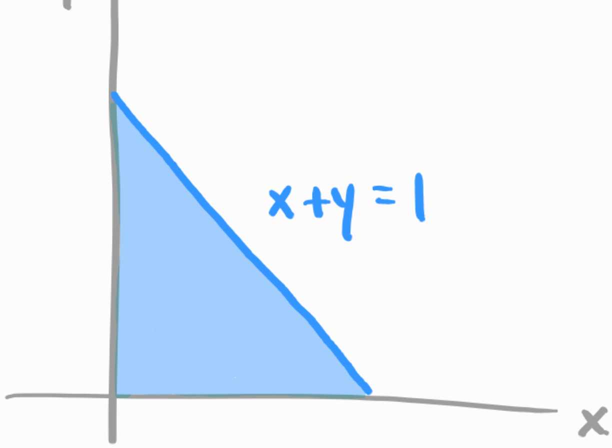
$$= \frac{27x^2}{4}\Big|_{x=0}^{x=1} = \frac{27}{4}.$$

$\boxed{\text{Ex}}$ Let's find the total mass of the region R bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$ whose density is given by $f(x,y,z)=z$.



From the figure, we can see that it's possible to integrate between $z=0$ and $z=1-x-y$ over the base triangle

y



This xy-plane region in turn can be parametrized by the bounds $0 \le x \le 1$ and $0 \le y \le 1-x$.

Here's the full iterated integral:

$$\iiint_R z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{}$$

$$= \int_0 \int_0 \frac{(1-x-y)^2}{2} \, dy \, dx$$

$$= \int_0^1 \left[-\frac{(1-x-y)^3}{6} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[-\frac{\cancelto{0}{(1-x-(1-x))^3}}{6} + \frac{(1-x)^3}{6} \right] dx$$

$$= \int_0^1 \frac{(1-x)^3}{6} \, dx$$

$$= -\frac{(1-x)^4}{24} \Bigg|_{x=0}^{x=1} = \frac{1}{24}.$$

Next time: more triple integrals.