Lecture 15.5

Last time:

- To compute certain triple integrals, it may be advantageous to convert to cylindrical coordinates:

$$(x, y, z) \rightsquigarrow (r, \theta, z) \text{ where}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$z = z$$

$$(r\cos\theta, r\sin\theta, z) \leftsquigarrow (r, \theta, z)$$

$$dV \rightsquigarrow r\,dz\,dr\,d\theta.$$

- In other situations, it might be preferable to use spherical coordinates:

$$(x, y, z) \rightsquigarrow (r, \theta, \phi) \text{ where}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\phi = \arccos\left(\frac{z}{r}\right)$$

$$(r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi) \rightsquigarrow (r, \theta, \phi)$$

$$dV \rightsquigarrow r^2 \sin\phi\, dr\, d\phi\, d\theta$$

Surface Area

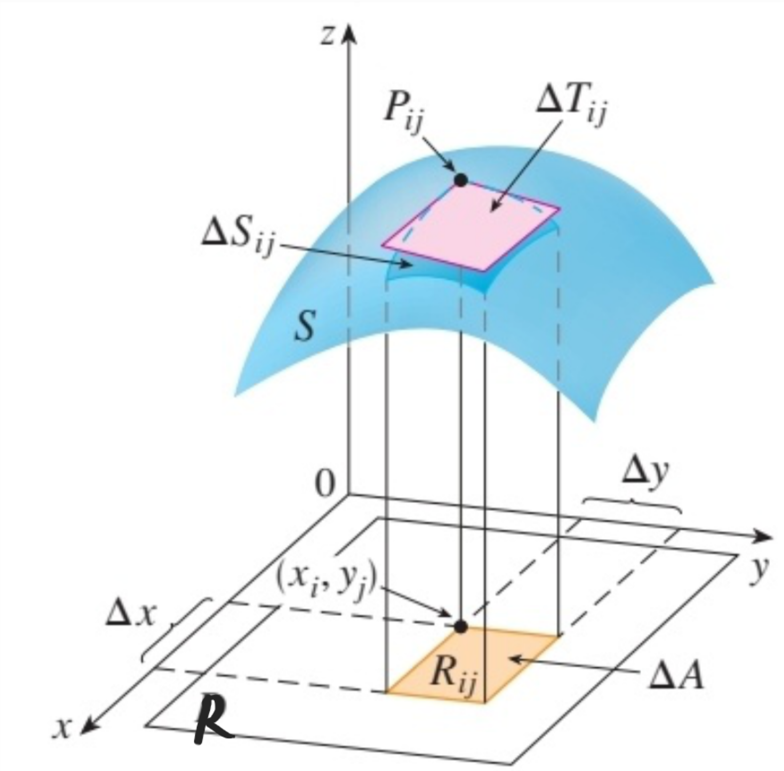
Q: How can we use integrals to find the surface area of a graph $z = f(x,y)$ over a certain region?

Key idea: the tangent plane

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

provides a good approximation to the

surface near (x_0, y_0, z_0), so the area of a small patch of the tangent plane gives a close approximation of the surface area of S = graph of f:



If we divide some base region R into small rectangles R_{ij} and choose points (x_i, y_j) in each rectangle, the surface area can be computed by

$$A(S \text{ over } R) = \lim_{m,n \to \infty} \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} \Delta T_{ij}$$

where ΔT_{ij} is the area of the small patch of the tangent plane at (x_i, y_j) lying over R_{ij}.

If we let $\vec{u}_{ij} = \langle 1, 0, f_x(x_i, y_j) \rangle \Delta x$

and $\vec{v}_{ij} = \langle 0, 1, f_y(x_i, y_j) \rangle \Delta y$

then $\Delta T_{ij} = |\vec{u}_{ij} \times \vec{v}_{ij}|$ which works

out to

$$\Delta T_{ij} = \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\ \Delta A$$

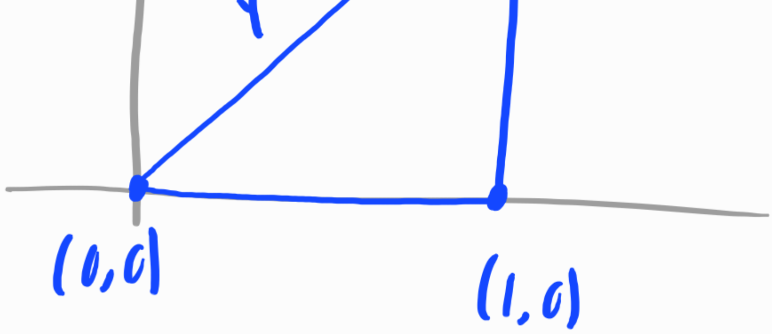
where $\Delta A = \Delta x \Delta y$.

Exercise 1: Verify this formula by computing the cross product above, or go find it in the textbook.

Taking the number of rectangles R_{ij} to ∞, we get:

$$\boxed{A(S \text{ over } R) = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA.}$$

Ex Let's find the surface area of $z = x^2 + 2y$ over the triangle R with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



The base region to integrate over has

bounds $0 \le x \le 1$ and $0 \le y \le x$,

so

$$A(S) = \int_0^1 \int_0^x \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx.$$

Here are the partial derivatives of

$f(x,y) = x^2 + 2y$:

$f_x = 2x$ and $f_y = 2$.

Then

$$A(S) = \int_0^1 \int_0^x \sqrt{4x^2+5}\; dy\, dx$$

$$= \int_0^1 \left[\sqrt{4x^2+5}\; y \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 x\sqrt{4x^2+5}\; dx$$

$$u = 4x^2+5$$

$$du = 8x\; dx$$

$$= \int_5^9 x \cdot u^{1/2} \cdot \frac{du}{8x}$$

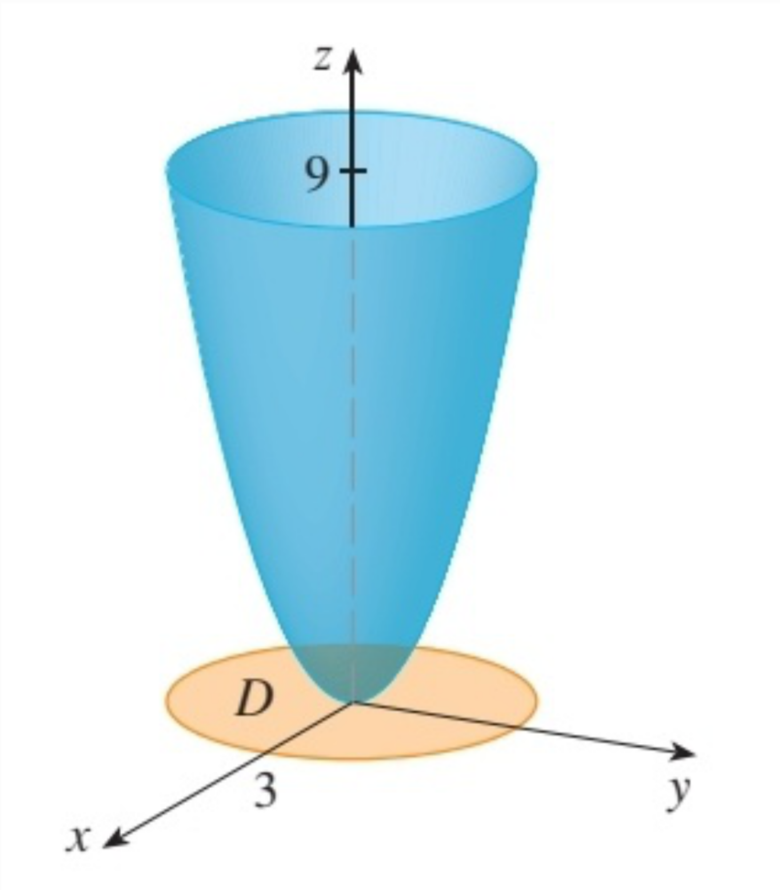
$$= \frac{1}{8} \int_5^9 u^{1/2}\, du$$

$$= \frac{1}{8} \left[\frac{2}{3} u^{3/2} \right]_{u=5}^{u=9}$$

$$= \frac{1}{12}\left(27 - 5^{3/2}\right).$$

$\boxed{\text{Ex}}$ Let's find the surface area of

$z = x^2 + y^2$ below $z = 9$.



The figure shows how to "project" down into the xy-plane and find a region to integrate over, but let's do it algebraically.

The graph of $z = x^2 + y^2$ has circular level curves

$$x^2 + y^2 = k$$

which grew as k (the height) increases.

There are no level curves for $k < 0$

and the largest one in the specified

range $(z \leq 9)$ is

$$x^2 + y^2 = 9$$

so let's integrate over this circle,

with bounds $-3 \leq x \leq 3$ and

$-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$.

Now for the integrand: if $f = x^2 + y^2$,

$f_x = 2x$ and $f_y = 2y$,

so

$$A(S) = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1} \, dy \, dx.$$

This looks like a job for polar coordinates:

$$A(S) = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1} \, r \, dr \, d\theta$$

make sure you can set this up yourself, or go practice it more

We can set $u = 4r^2+1$ and $du = 8r\,dr$,

$$\int r\sqrt{4r^2+1}\,dr = \int r\,u^{1/2} \cdot \frac{du}{8r}$$

$$= \frac{1}{8}\int u^{1/2}\,du$$

$$= \frac{1}{12}u^{3/2} + c$$

$$= \frac{1}{12}(4r^2+1)^{3/2} + c$$

So $A(S) = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\; r\,dr\,d\theta$

$$= \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^{r=3} d\theta$$

$$= \frac{1}{\ }\int_0^{2\pi}\left(37^{3/2} - 1\right) d\theta$$

$$= \frac{37^{3/2}-1}{12} \theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{(37^{3/2}-1)\pi}{6}.$$

Exercise 2: Compute the surface area of

(a) A cylinder with radius R and height h.

(b) A sphere of radius R.

(c) The patch of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

Next time: vector-valued functions.