$$\boxed{\text{Lecture 15.6}}$$

Last time:

- The mass of an object can be computed by

$$\text{mass} = \text{density} \times \text{size}.$$

- When R is a region in $\mathbb{R}^2$ with density function $\rho(x,y)$,

$$\text{Mass}(R) = \iint_R \rho(x,y)\, dA.$$

- When R is a region in $\mathbb{R}^3$ with density function $\rho(x,y,z)$,

$$\text{Mass}(R) = \iiint_R \rho(x,y,z)\,dV$$

- For $R = [a,b] \times [c,d] \times [e,f]$,

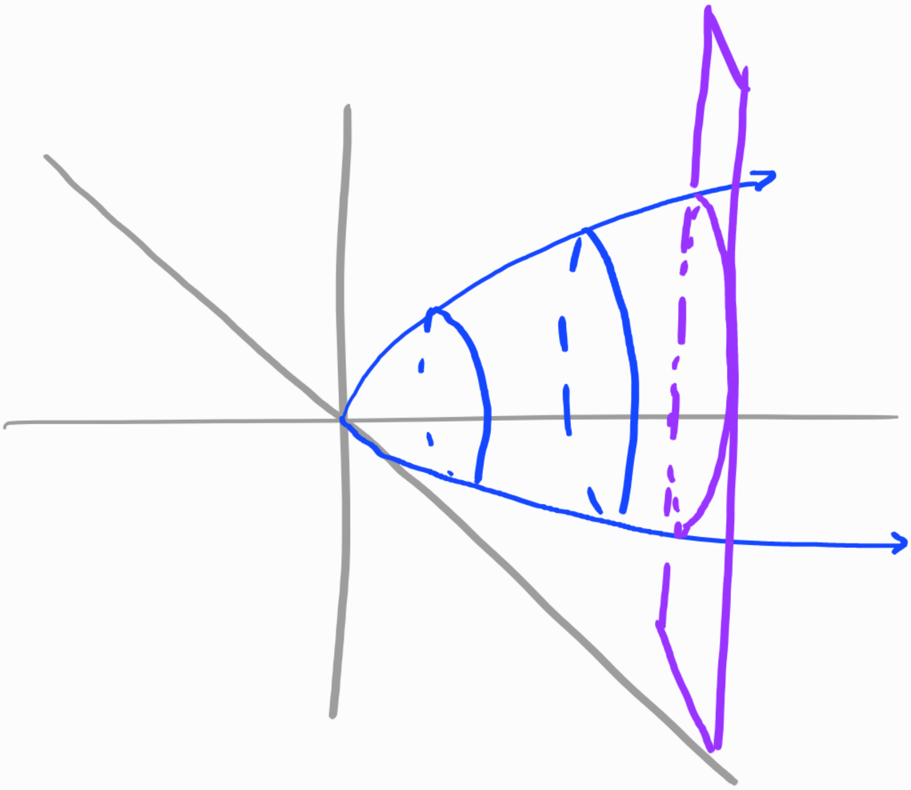
$$\iiint_R \rho(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f \rho(x,y,z)\,dz\,dy\,dx.$$

$\boxed{\text{Ex}}$ Let's find the mass of the region R bounded by $y = x^2 + z^2$ and $y = 4$, with (a) constant density 1

(this computes vol(R)!)

(b) density $f(x,y,z) = \sqrt{x^2+z^2}$.

Here's a sketch of R:



We can "see" that R is bounded by

$$x^2+z^2 \le y \le 4$$

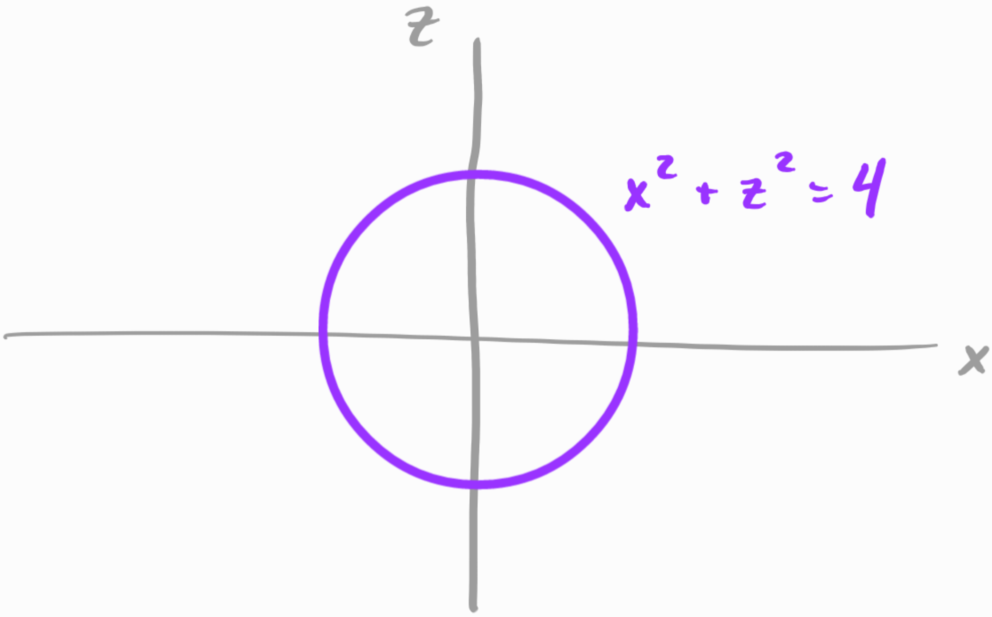
so let's put dy on the innermost

integral and draw a sketch of the "base region" in the xz-plane.

The cross section $y = 0$ just gives

$$x^2 + z^2 = 0 \implies (x,z) = (0,0).$$

We should instead use the "largest cross section" at $y = 4$:



This has top and bottom functions

$z = \pm\sqrt{4 - x^2}$ so we can form

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our triple integrals

(a) $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} 1 \, dy\, dz\, dx$$

(b) $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2} \, dy\, dz\, dx$$

Exercise 1: Compute both integrals.

Hint: use polar coordinates once you're down to a double integral, or wait for cylindrical coordinates below.

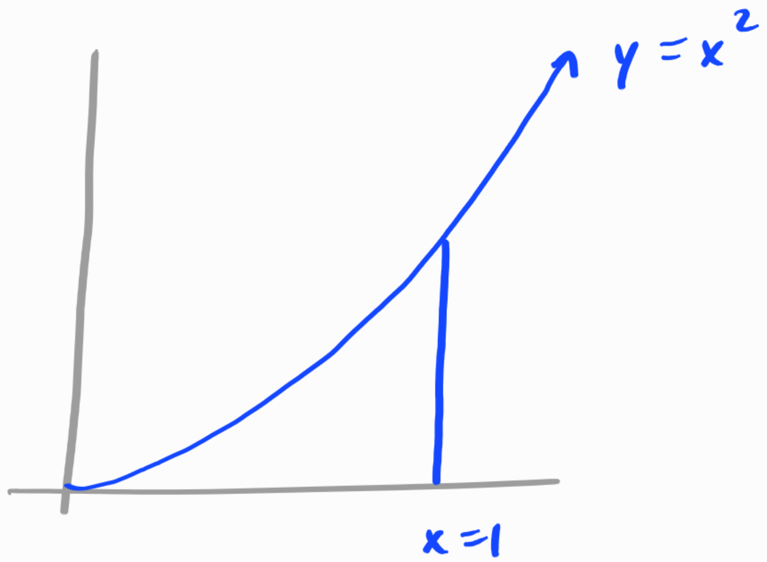
$\boxed{\text{Ex}}$ How do we switch the order in

$$\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\, dz\, dy\, dx \ ?$$

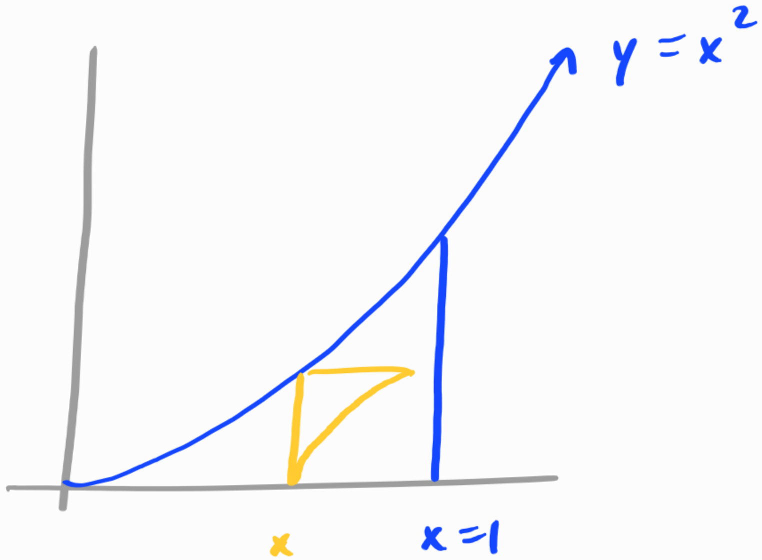
Starting with the two outer integrals

$$\int_0^1 \int_0^{x^2} V(x,y)\, dy\, dx$$

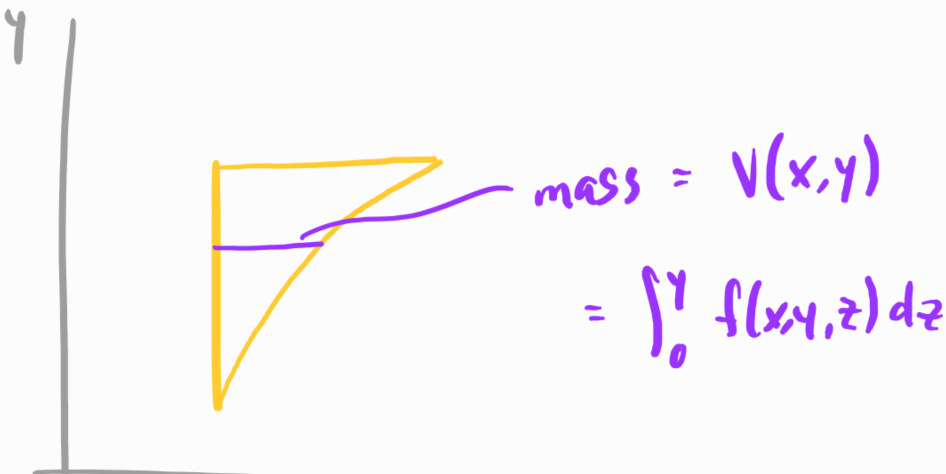
we can sketch this region in the xy-plane:



In the triple integral, we are first scanning in the x-direction and picking out a cross section in the y-direction, which looks like



But for each $0 \le y \le x^2$, we're further integrating a cross section (to get a mass):

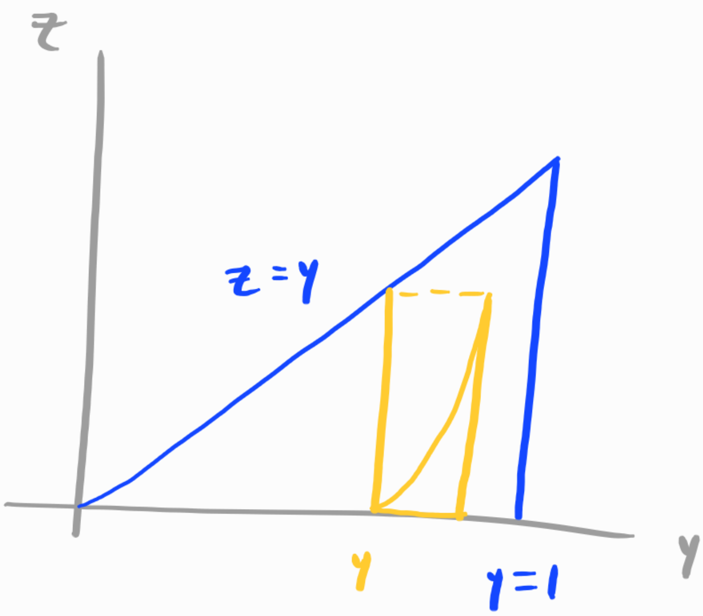


Let's change the order to $dx\,dz\,dy$.

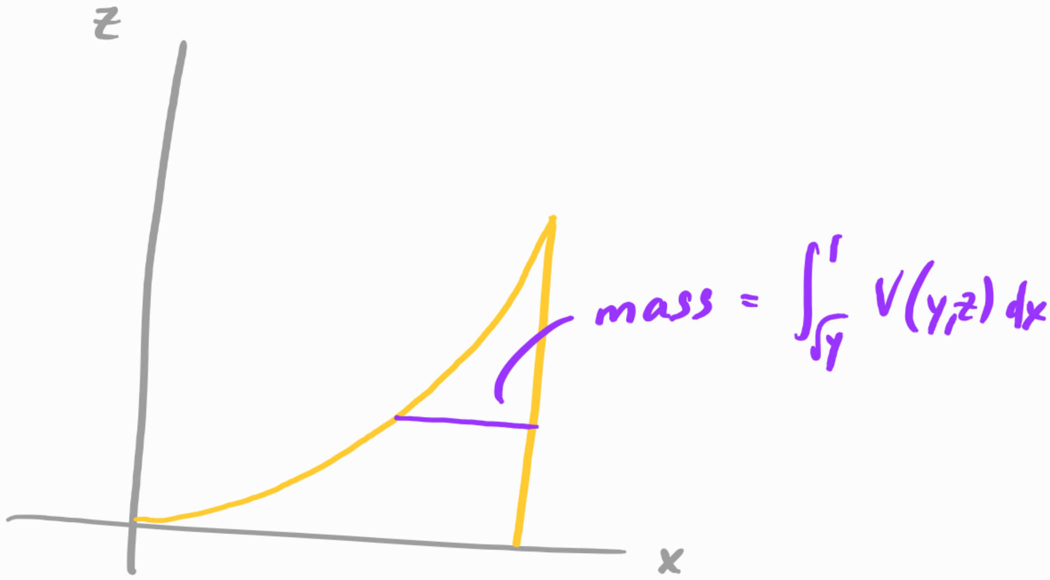
In the yz-plane, y is now bounded between 0 and 1, and z is between

look at the xy-plane figure

0 and y:



One cross section at y looks like



Then the triple integral can be written:

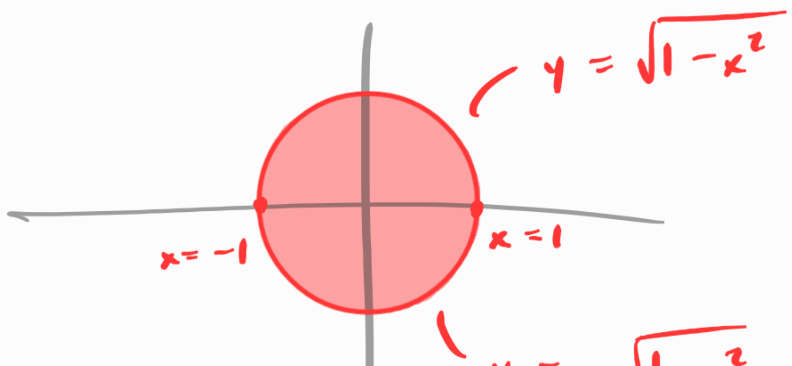
$$\iiint_R f(x,y,z)\,dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy.$$

$\boxed{\text{Ex}}$ Let's find the volume of the region R enclosed by $x^2+y^2=1$, below $z=4$ and above $z = 1-x^2-y^2$ using

$$\iiint_R 1 \, dV.$$

Since $1 - x^2 - y^2 \leq z \leq 4$ gives natural z-bounds, let's integrate z on the inside.

This leaves x and y to handle, but the image of x^2+y^2+1 (a cylinder) in the xy-plane is just the unit circle:



$y = -\sqrt{1-x^2}$

So the triple integral is

$$\mathrm{vol}(R) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-x^2-y^2}^{4} 1 \, dz\, dy\, dx$$

which doesn't look particularly nice.

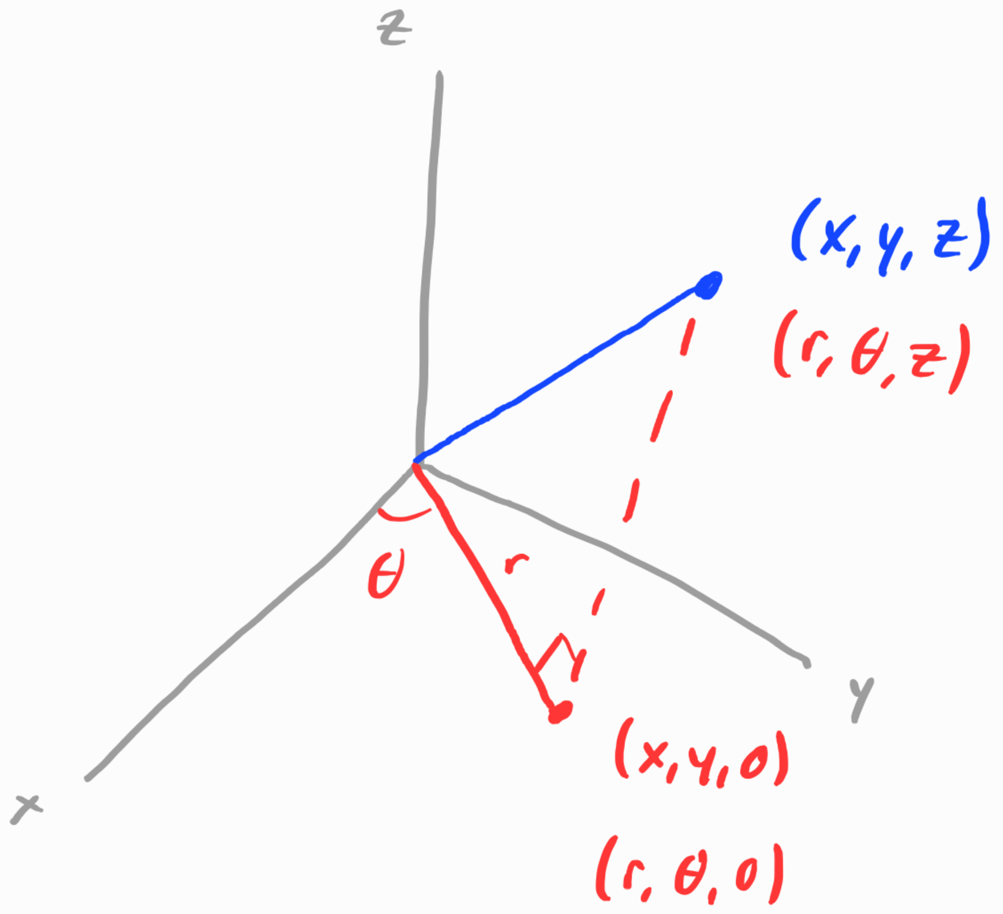
As we did with polar coordinates in $\mathbb{R}^2$, there's an alternative coordinate in $\mathbb{R}^3$ we can use to reduce the complexity of this integral.

Def The *cylindrical coordinate system* in $\mathbb{R}^3$ has coordinates (r, θ, z) where

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$z = z.$$



In words, cylindrical coordinates are

"polar coordinates plus height".

To integrate in cylindrical coordinates, use the substitution

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

In our example, the x- and y-bounds can be rewritten in polar coordinates:

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

while $z = 1 - x^2 - y^2$ becomes

$z = 1 - r^2$

Then

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-x^2-y^2}^{4} 1\, dz\,dy\,dx$$

$$= \int_{0}^{2\pi}\int_{0}^{1}\int_{1-r^2}^{4} r\, dz\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1}\Big[rz\Big]_{z=1-r^2}^{z=4} dr\,d\theta$$

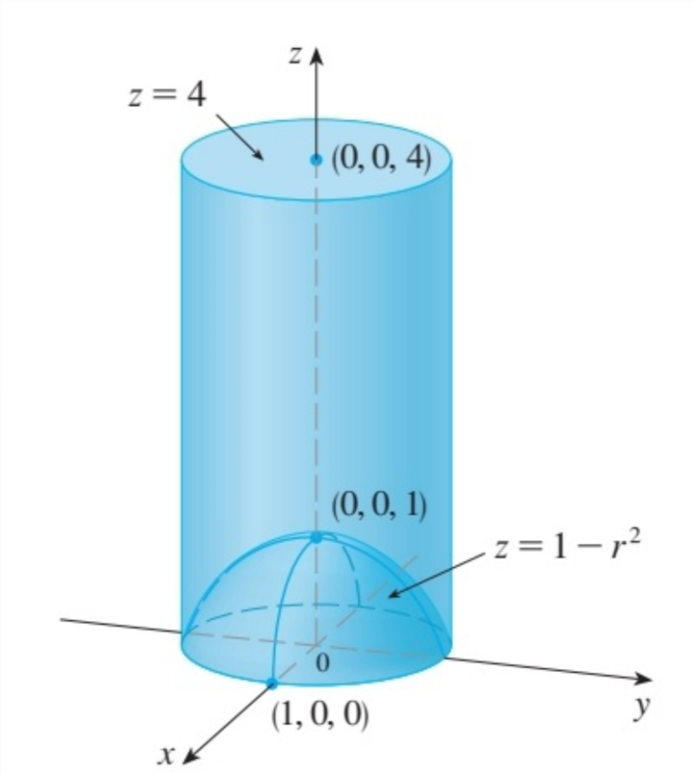
$$= \int_{0}^{2\pi}\int_{0}^{1}\left(4r - r^2 + r^3\right) dr\,d\theta$$

$$= \int_{0}^{2\pi}\left[2r^2 - \frac{1}{3}r^3 + \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \int_{0}^{2\pi}\left(2 - \frac{1}{3} + \frac{1}{4}\right) d\theta = \int_{0}^{2\pi}\frac{23}{24}\, d\theta$$

$$= \frac{23}{24}\,\theta\,\Big|_0^{2\pi} = \frac{23\pi}{12}.$$

Here's a graph of the region whose volume we just computed:

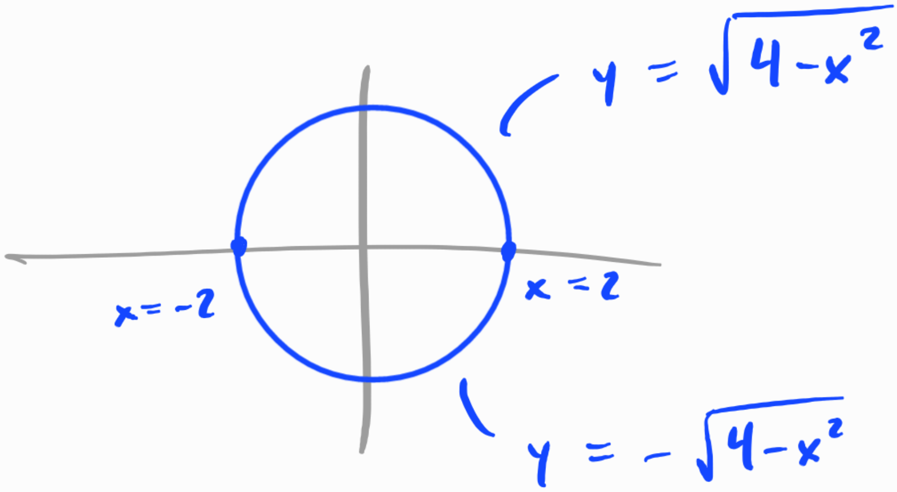


Ex Let's compute

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\,dy\,dx$$

using cylindrical coordinates.

On the outside, we have



This is captured by

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

while

$$z = \sqrt{x^2+y^2} = r$$

$$f(x,y,z) = x^2+y^2 = r^2.$$

Then the integral becomes

$$\int_0^{2\pi}\int_0^{2}\int_r^{4} r^2 \cdot r\,dz\,dr\,d\theta.$$

Exercise 2: Finish this problem by computing the triple integral.

Exercise 3: Compute each of the following regions' volumes or, if there's a density function $f(x,y,z)$ given, their masses.

(a) R is enclosed by $y = 4x^2 + z^2$

and $y = 4$.

(b) R is enclosed by $y = \sqrt{25 - x^2}$,

$z = 6 - y$ and $z = \sqrt{y}$.

(c) $R = [0, 10] \times [0, 10] \times [0, 10]$,

$$f(x, y, z) = \begin{cases} \frac{1}{V}, & (x, y, z) \text{ is in } R \\ 0, & (x, y, z) \text{ is outside } R \end{cases}$$

where $V = \mathrm{vol}(R)$.

(d) R is above the xy-plane, below

the cylinder $x^2 + z^2 = 16$ and between $y = 0$ and $y = 3$; $f(x,y,z) = 4yz$.

Note: Another useful coordinate system in $\mathbb{R}^3$ is the spherical coordinate system:

$$(x, y, z) \rightsquigarrow (r, \theta, \phi) \text{ where}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$(x, y, z) \leftsquigarrow (r, \theta, \phi)$$

where

$$x = r\cos\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\phi$$

The differential changes by

$$\boxed{dV = r^2\sin\phi \, dr\, d\theta\, d\phi}$$

Exercise 4: Compute

$$\iiint_R e^{\sqrt{x^2+y^2+z^2}} \, dV$$

where R is the region enclosed

by $x^2+y^2+z^2 \leq 1$.

Next time: surface area.