Lecture 16.1

Last time:

- A parametric curve C is vector valued function of one variable.

$$C: \ r(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle.$$

- The arc length of C from $t=a$ to $t=b$ is

$$L = \int_a^b |r'(t)|\, dt.$$

Path Integrals

For a parametric curve C given by

$$r(t) = \langle x_1(t), \ldots, x_n(t) \rangle,$$

set $ds = |r'(t)|\, dt$

$$= \sqrt{x_1'(t)^2 + \ldots + x_n'(t)^2}\, dt.$$

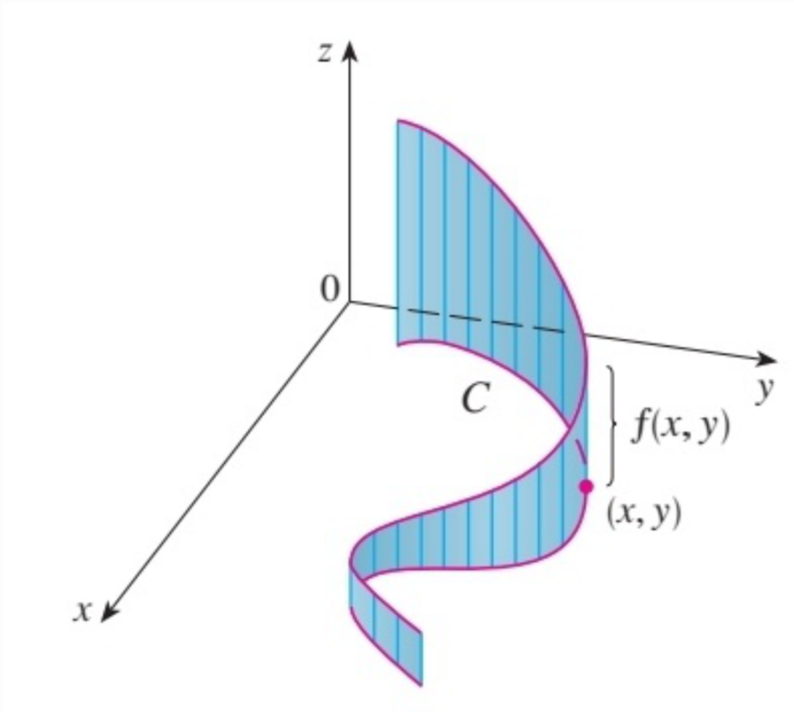
Then $L = \int_a^b 1\, ds$. What happens if we replace 1 with a "density function" $f(x_1, \ldots, x_n)$?

$\boxed{\text{Def}}$ The path integral (or line integral) along $C: r(t) = \langle x_1(t), \dots, x_n(t) \rangle$ from

$t = a$ to $t = b$ of a function $f(x_1, \dots, x_n)$

is

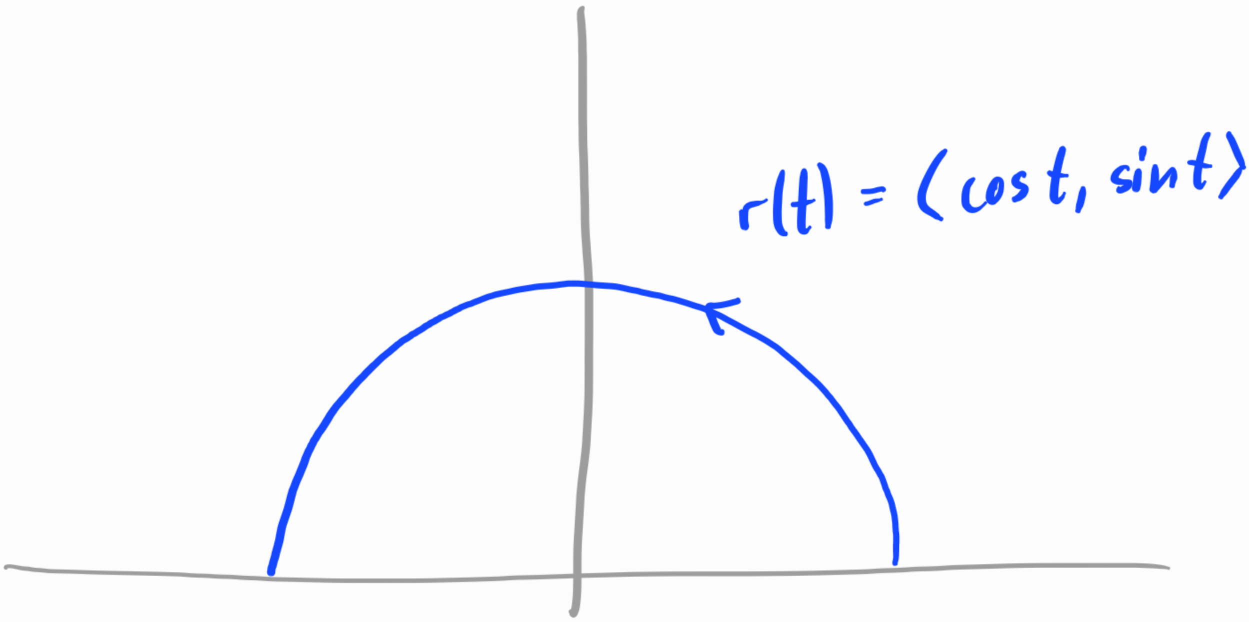
$$\underbrace{\int_C f \, ds}_{\text{mass}} = \int_a^b \underbrace{f(r(t))}_{\text{density}} \underbrace{|r'(t)| dt}_{\text{length}}.$$



$\boxed{\text{Ex}}$ Let's integrate $f(x,y) = 2 + x^2 y$

along the upper half of $x^2+y^2=1$,

oriented counterclockwise.



The curve in question, say C, is

parametrized by

$$r(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi.$$

The differential ds is

$$ds = \sqrt{(-\sin t)^2 + \cos^2 t}\, dt = dt.$$

Then the path integral is

$$\int_C (2 + x^2 y)\, ds = \int_0^{\pi} (2 + \cos^2 t \sin t)\, dt$$

$$= \left[2t - \frac{1}{3}\cos^3 t \right]_0^{\pi}$$

$$= 2\pi + \frac{1}{3} - \left(0 - \frac{1}{3}\right)$$

$$= 2\pi + \frac{2}{3}.$$

Note: for $\int_C f\, ds$ to exist (i.e.

for the corresponding limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*)) \Delta s_i$$

to exist), it is necessary for

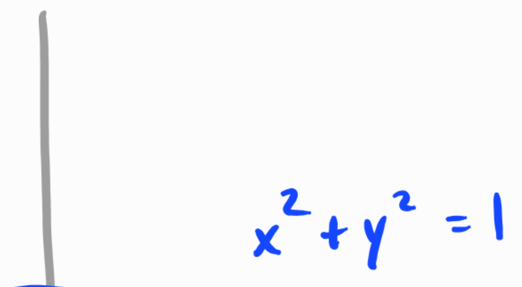
C to be a smooth curve:

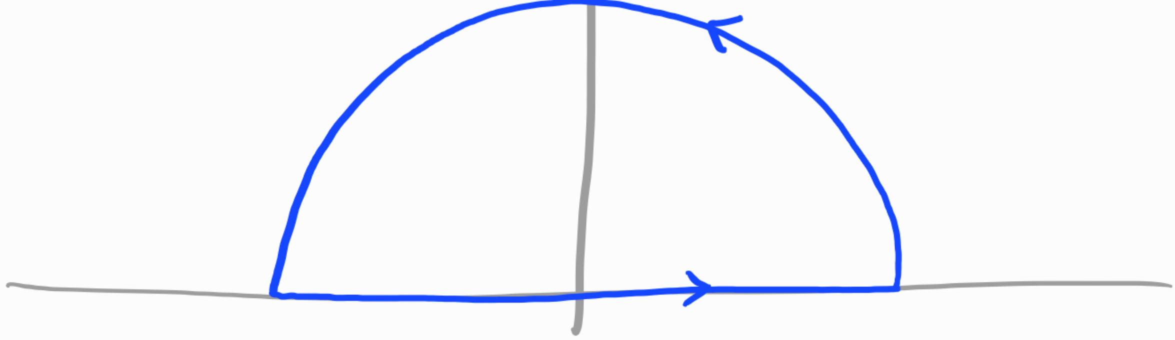
$$r'(t) = 0 \quad \text{for all} \quad a \le t \le b$$

and f to be continuous.

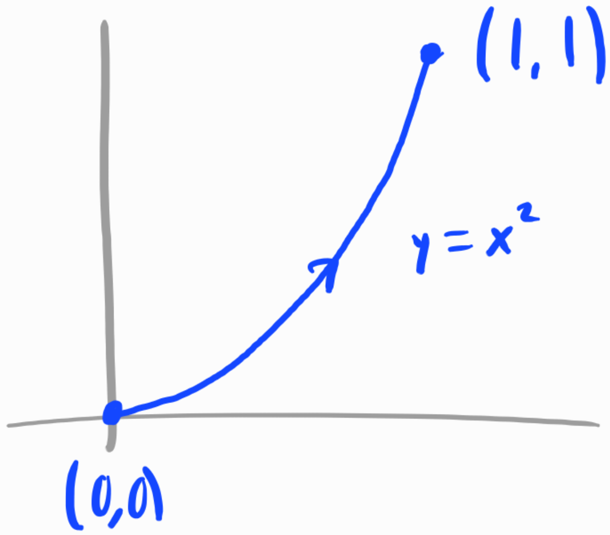
Exercise 1: Integrate $f(x,y) = 2 + x^2 y$

along





Exercise 2: Find the center of mass of the thin wire



if the density of the wire is given by $f(x,y) = 2x$.

Note: The orientation of C does not change the value of a path integral along C:

$$\int_b^a f\,ds = \int_a^b f\,ds.$$

This is because ds only depends on $x'(t)^2$ and $y'(t)^2$.

Think of a density function $f(x_1, \dots, x_n)$ as assigning a single value (the output) at every point along C.

Q: What if we attach a vector at every point?

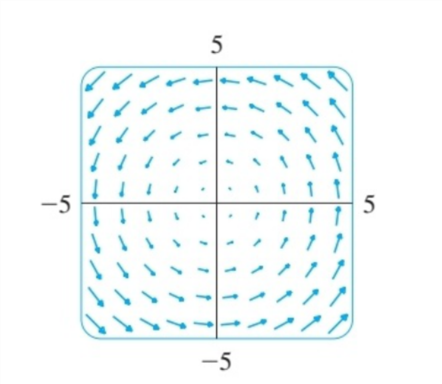
Vector Fields

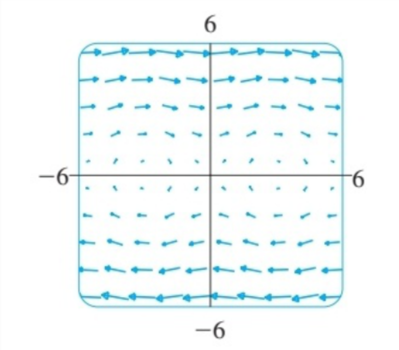
Def A **vector field** in $\mathbb{R}^n$ is a vector valued function with n inputs and n outputs:

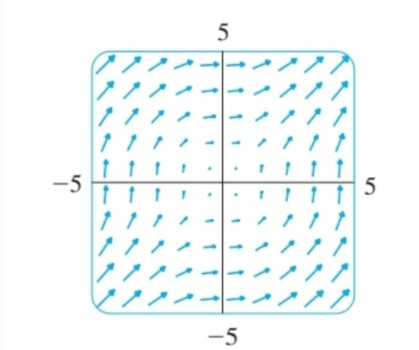
$$F(x_1,\dots,x_n) = \langle F_1(x_1,\dots,x_n),\dots,F_n(x_1,\dots,x_n)\rangle.$$

We can visualize this as the assignment of an n-dimensional vector to each point in $\mathbb{R}^n$

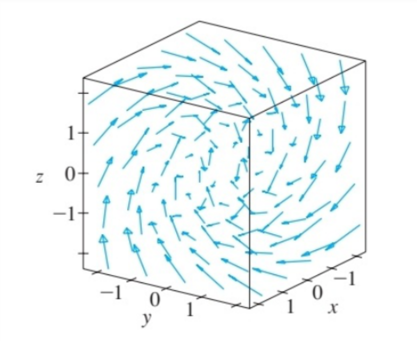
in $\mathbb{R}^n$:

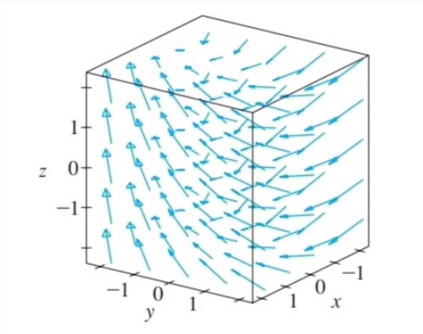


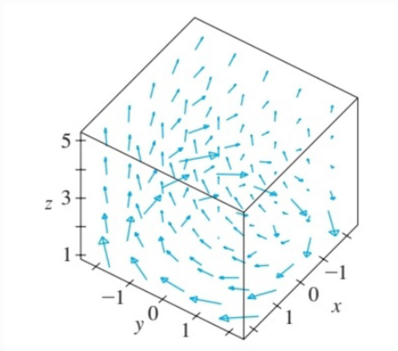




$F = \quad \langle -y, x \rangle \qquad \langle y, \sin x \rangle \qquad \langle \ln(1+y^2), \ln(1+x^2) \rangle$

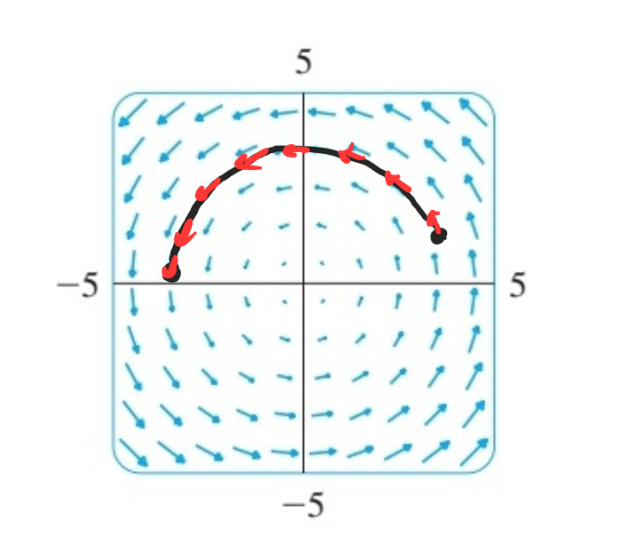






$F = \quad \langle y, z, x \rangle \qquad \langle y, -z, x \rangle \qquad \left\langle \frac{y}{z}, \frac{-x}{z}, \frac{z}{4} \right\rangle$

Conceptually, we can think of a vector field F as a force pushing particles around in n-dimensional space:



Ex One of the most important types of vector fields is a gradient field:

$$\underline{F} = \nabla \underline{f}$$ for a differentiable function $f(x_1, \ldots, x_n)$.

vector field

scalar function

In general, a vector field F is called **conservative** if $F = \nabla f$ for some f.

$\boxed{\text{Def}}$ The **path integral** (or **line integral**) of a continuous vector field F along a smooth curve $C: r(t),\ a \leq t \leq b$ is

$$\int_C F \cdot dr = \int_C F \cdot T\, ds$$

$$= \int_a^b f(r(t)) \cdot r(t)\, dt$$

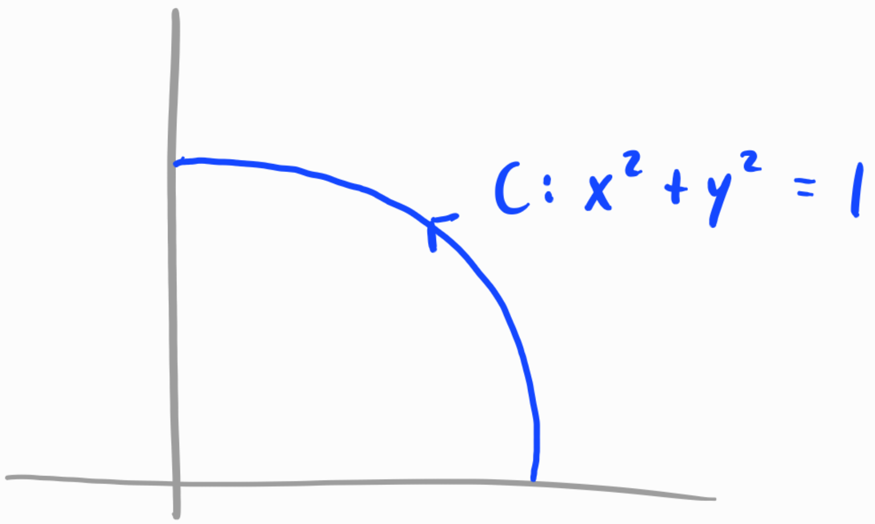
where $T(t) = \dfrac{r'(t)}{|r'(t)|}$ is the unit tangent vector function for $r(t)$.

Interpretation: F represents a force acting on particles in $\mathbb{R}^n$ — we can visualize this as a physical force pushing the particles, but the situation could be more abstract, e.g. electromagnetic forces acting on electric charge, velocity of a fluid flowing through space, gravity, etc.

$\boxed{\text{Ex}}$ If F is a physical force, the work performed by F on a particle is computed by

$$\text{work} = \text{force} \times \text{distance}.$$

For $F(x,y) = \langle x^2, -xy \rangle$ acting on a particle traveling along this quarter circle,



the total work performed by F along

this path is

$$W = \int_C \underbrace{F}_{\text{force}} \cdot \underbrace{dr}_{\text{distance}}.$$

To compute W, we parametrize C by

$$C: r(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi/2$$

and use $r'(t) = \langle -\sin t, \cos t \rangle$:

$$W = \int_C F \cdot dr$$

$$= \int_0^{\pi/2} \langle x^2, -xy \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$x = \cos t$

$y = \sin t$

$$= \int_0^{\pi/2} \left(-\cos^2 t \sin t - \cos t \sin t \cos t\right) dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \, dt$$

$$= \left[\frac{2}{3}\cos^3 t\right]_0^{\pi/2} = -\frac{2}{3}.$$

Warning: Although we saw that for a scalar function $f(x_1, \ldots, x_n)$,

$\int_C f \, ds$ does not depend on the

orientation of C, the path integral of a vector field $F(x_1, \dots, x_n)$ along a curve C does depend on orientation:

$$\int_b^a F \cdot T \, ds = -\int_a^b F \cdot T \, ds.$$

Exercise 3: Verify this by orienting the quarter circle in the previous example in the opposite direction and computing $\int_C F \cdot dr$ again. Interpret this.

Q: What happens if we integrate a conservative vector field $F = \nabla f$ along a curve?

$\boxed{\text{Fundamental Theorem of Line Integrals}}$

Let $C: r(t),\ a \leq t \leq b$ be a smooth curve. Then for any conservative vector field $F = \nabla f$ which is continuous along C,

$$\int_C F \cdot dr = f(r(b)) - f(r(a)).$$

Note: Compare this to the Fundamental Theorem of Calculus.

Exercise 3: Compute the following path integrals. In each, f is a scalar function and F is a vector field.

(a) $\int_C F \cdot dr$ where $F = \langle 8x^2yz, 5z, -4xy \rangle$

and $C: \ r(t) = \langle t, t^2, t^3 \rangle, \ 0 \le t \le 1.$

(b) $\int_C F \cdot dr$ where $F = \langle xz, 0, -yz \rangle$

and C is the line segment from

$(-1, 2, 0)$ to $(3, 0, 1)$.

(c) $\int_C \nabla f \cdot dr$ where

$$f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$$

and C is the line segment from

$(1, \frac{1}{2}, 2)$ to $(2, 1, -1)$. What would happen if you chose a different path from $(1, \frac{1}{2}, 2)$ to $(2, 1, -1)$?

(d) $\int_C \nabla f \cdot dr$ where $f = x^3(3-y^2) + 4y$

and $C: r(t) = \langle 3-t^2, 5-t \rangle, -2 \leq t \leq 3$.

What happens if you choose a different path with the same endpoints?

Next time: more path integrals.