Lecture 16.3

Last time:

- The **path integral** of a single function $f(x_1, \dots, x_n)$ along a parametric curve $C: r(t)$, $a \le t \le b$, is

$$\int_C f \, ds = \int_a^b f(r(t)) \, |r'(t)| \, dt.$$

- A **vector field** in $\mathbb{R}^n$ is a vector valued function with n inputs and n outputs: $F(x_1, \dots, x_n) = \langle F_1, \dots, F_n \rangle$.

- The path integral of a vector field F along a curve $C: r(t)$, $a \le t \le b$, is

$$\int_C F \cdot dr = \int_C F \cdot T \, ds = \int_a^b F(r(t)) \cdot r'(t) \, dt.$$

This computes the work (= force × distance) performed by F on a particle traveling along C.

- A vector field F is conservative if $F = \nabla f$ for some $f(x_1, \ldots, x_n)$.

- $\boxed{\text{Fundamental Theorem of Line Integrals}}$

If $F = \nabla f$ is a conservative vector field and $C: r(t)$, $a \le t \le b$, is a smooth curve along which F is continuous,

$$\int_C F \cdot dr = f(r(b)) - f(r(a)).$$

$\boxed{\text{Ex}}$ The force of gravity acting on an object at position $\vec{x} = \langle x, y, z \rangle$ and mass m relative to an object of mass

M at $(0,0,0)$ is

$$F(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$

where $G \approx 6.67 \times 10^{-11}$ is the gravitational constant.

One can check that $F = \nabla f$ where

$$f(\vec{x}) = \frac{mMG}{|\vec{x}|^2}.$$

If we want to compute the work done by gravity on an object traveling from

$(3,4,12)$ to $(2,2,0)$, the **Fundamental Theorem of Line Integrals** says that we just need to pick a path C given by $r(t)$, $a \le t \le b$, with $r(a) = (3,4,12)$ and $r(b) = (2,2,0)$. Then

$$\text{work} = \int_C F \cdot dr = f(r(b)) - f(r(a))$$

$$= \frac{mMG}{|\langle 3,4,12\rangle|^2} - \frac{mMG}{|\langle 2,2,0\rangle|^2}$$

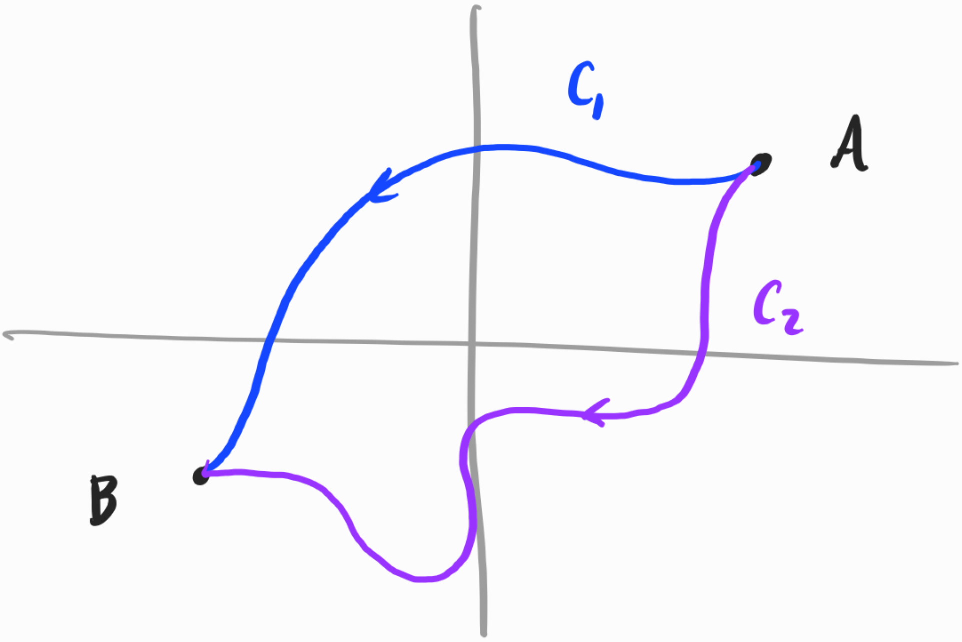
$$= mMG\left(\frac{1}{13} - \frac{1}{2\sqrt{2}}\right).$$

Note: We didn't actually pick a path C to calculate work in the last example. This is one of the main features of a conservative vector field:

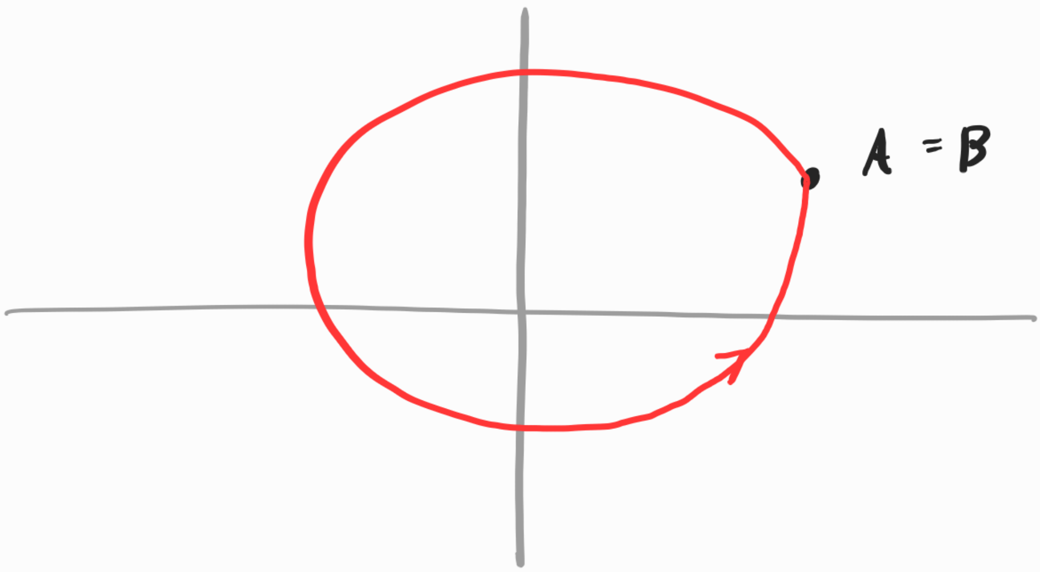
$\boxed{\text{Corollary}}$ Let F be a continuous, conservative vector field.

(1) If C_1 and C_2 are smooth paths with the same endpoints, then

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr.$$



(2) If C is a closed path, meaning it starts and ends at the same point, then $\int_C F \cdot dr = 0$.



(1) says $\int_C \mathbf{F} \cdot d\mathbf{r}$ is "independent of path".

(2) explains why "conservative" is the right name for these vector fields: work is conserved along any closed path in space.

In fact, (1) and (2) are equivalent; see the textbook for details.

$\boxed{\text{Theorem}}$ If $\mathbf{F}$ is a continuous vector field on an open region D in $\mathbb{R}^n$

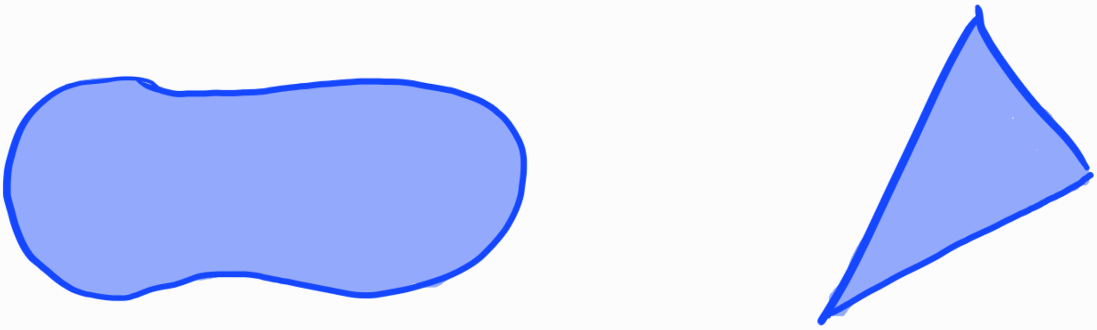
opposite of closed: none of the boundary points are in the region and if $\int_C F \cdot dr$ is independent of path for any C in D, then F is conservative.

Q: How can we tell if a vector field F is conservative?

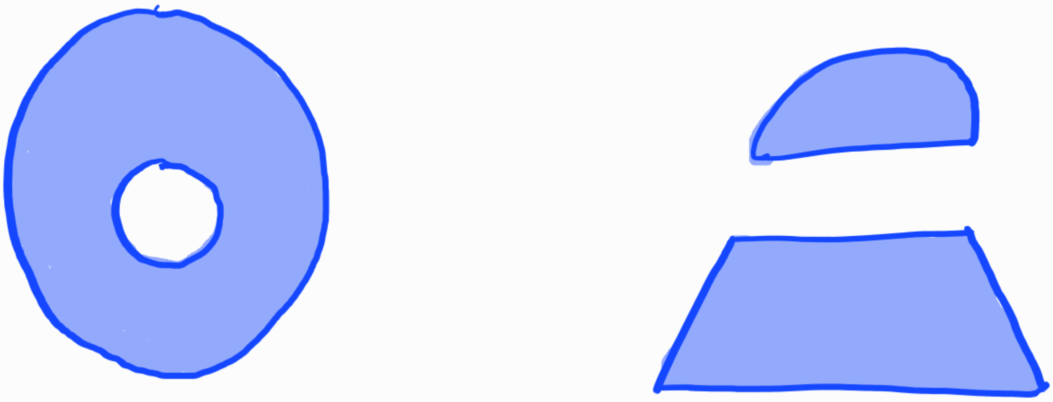
$\boxed{\text{Def}}$ A **simply connected region** is a region D in $\mathbb{R}^n$ with the property that any simple closed curve C in D encloses a

C does not intersect itself

subregion of D.



simply connected



not simply connected

Theorem Let $F = \langle P, Q \rangle$ be a vector field on a region D in $\mathbb{R}^2$ such that $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial y}$ are continuous on D.

(1) If F is conservative, then

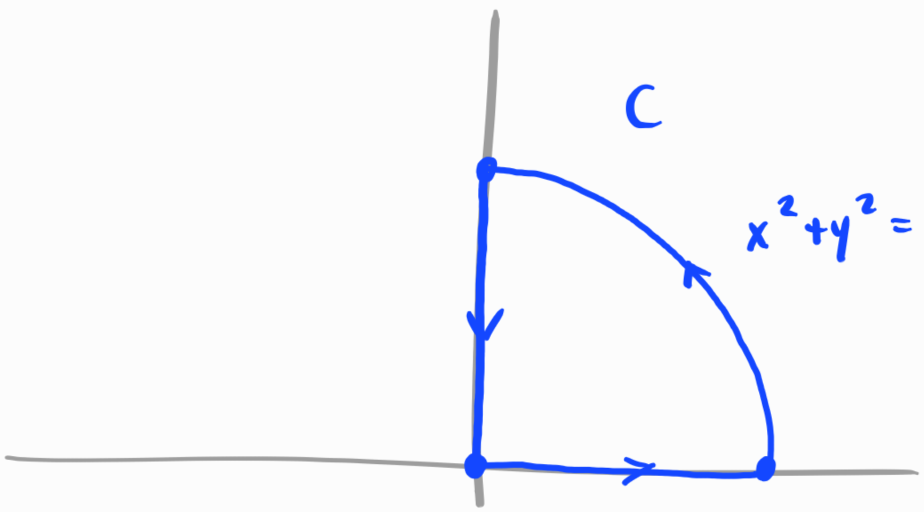
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(2) If D is simply connected and open and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then F is conservative.

$\boxed{\text{Ex}}$ Let's use the properties above to compute $\int_C F \cdot dr$ where

$$F = \langle 3 + 2xy,\ x^2 - 3y^2 \rangle$$

and C is the curve



We could parametrize C (in 3 parts)

and compute $\int_C F \cdot dr$ directly, but if we can show F is conservative, the answer will be

$$\int_C F \cdot dr = 0.$$

Choose D to be an open disk containing the entire figure above (e.g. D could be the circle $x^2 + y^2 < 2$).

Then F has continuous partial derivatives on D, including

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3 + 2xy) = 2x$$

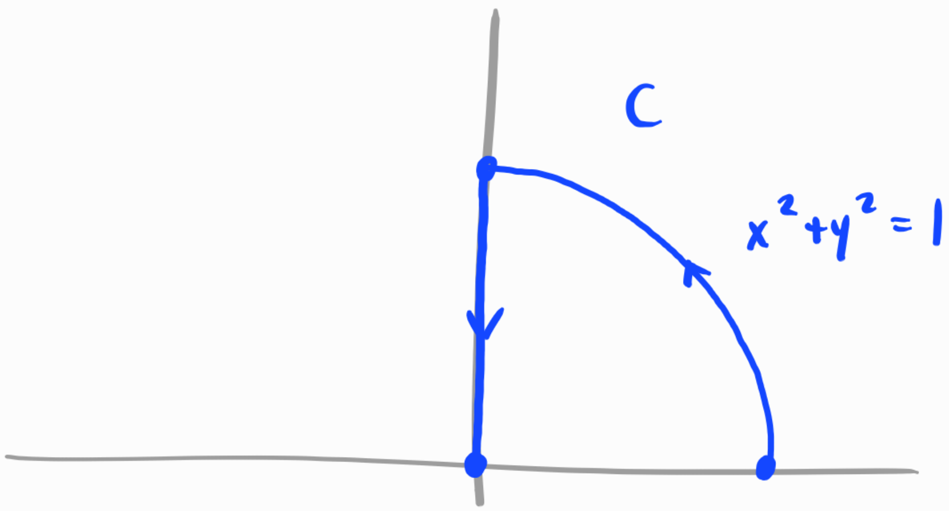
$$//$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x$$

so by the last Theorem, F is a conservative vector field.

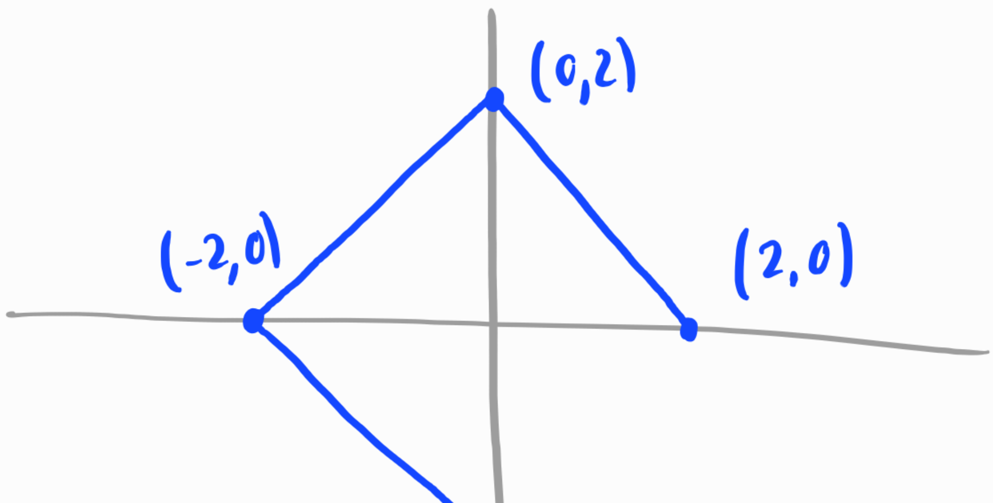
This implies $\int_C F \cdot dr = 0$ since C is closed.

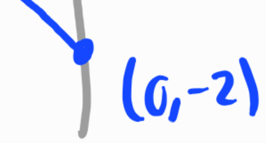
Exercise 1: Compute $\int_C F \cdot dr$ where F is the same as in the example above and C is the path



Exercise 2: Compute $\int_C F \cdot dr$ where

$F = \langle xe^{xy} - 2x,\ ye^{xy} + 3y^2 \rangle$ and C is





Note: When the path C is not closed, we actually need to find a function f with $\nabla f = F$ in order to use the Fundamental Theorem.

Exercise 3: Assuming each F is conservative, find f with $F = \nabla f$.

(a) $F = \langle 9x^2 - 3x^2y^2,\ 4 - 2x^3y \rangle$

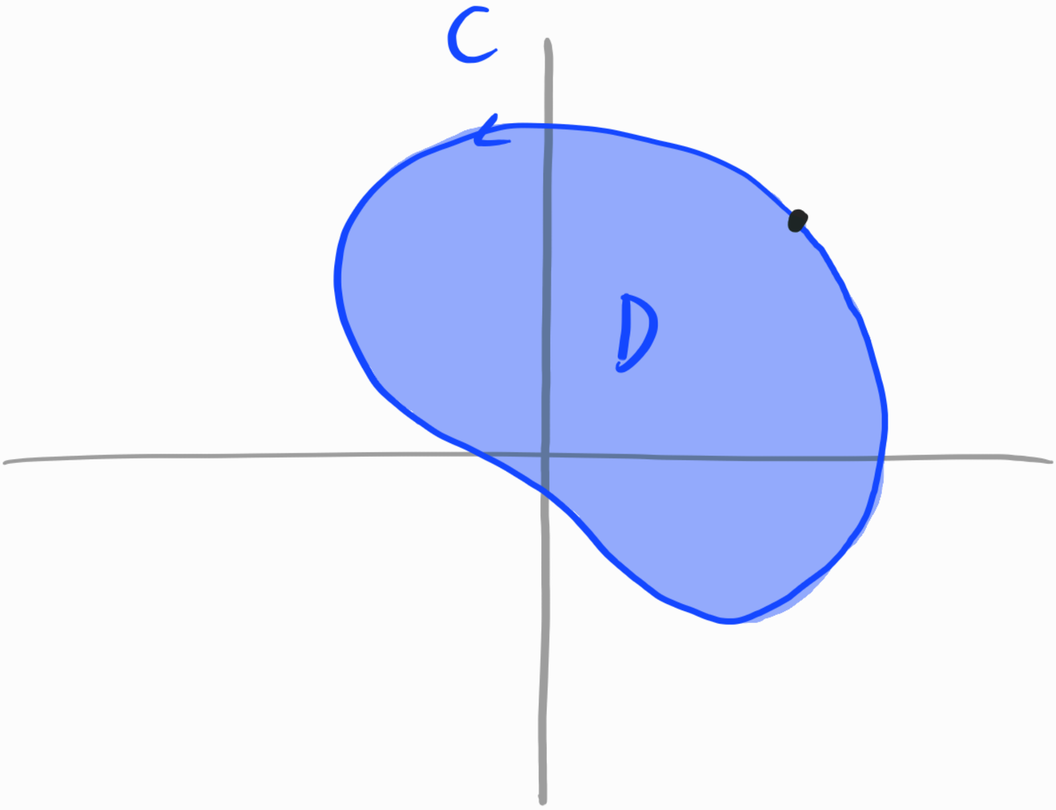
(b) $F = \left\langle 2xye^{x^2-1} + 4\sqrt{y},\ e^{x^2-1} + \frac{2x}{\sqrt{y}} \right\rangle$

Green's Theorem

Even if $F = \langle P, Q \rangle$ is not a conservative vector field, with

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

at some point on the region D, we can still use these partial derivatives to measure work along the boundary of D:



$\boxed{\text{Green's Theorem}}$ If C is a smooth, simple closed curve in $\mathbb{R}^2$ oriented counterclockwise (called "positively oriented"), then for any vector field $F = \langle P, Q \rangle$ with continuous partial derivatives,

$$\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where D is the region enclosed by C.

Next time: more on Green's Theorem.