Lecture 16.4

Last time:

- For a conservative vector field $F = \nabla f$,

$$\int_C F \cdot dr = f(r(b)) - f(r(a)).$$

- In particular, $\int_C F \cdot dr$ is independent of path C.

- On a simply connected region D in $\mathbb{R}^2$, $F = \langle P, Q \rangle$ being conservative is equivalent to

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

- When D is bounded by a simple closed curve C,

$$\int_C F \cdot dr = 0.$$

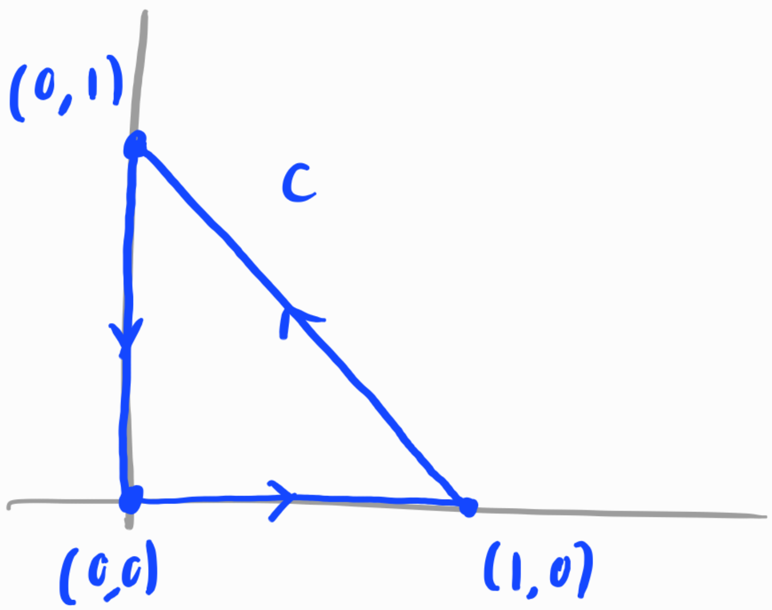
- $\boxed{\text{Green's Theorem}}$ For any vector field $F = \langle P, Q \rangle$ on a simply connected region D bounded by C, positively oriented,

$$\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Ex Let's find $\int_C F \cdot dr$ where

$F = \langle x^4, xy \rangle$ and C is the

perimeter of the triangle



Strategy 1 : Parametrize each side of

the triangle, set up 3 line integrals

and solve.

Strategy 2 (better): Use Green's Theorem!

$$P = x^4 \rightsquigarrow \frac{\partial P}{\partial y} = 0$$

$$Q = xy \rightsquigarrow \frac{\partial Q}{\partial x} = y$$

and D is the interior of the triangle, which is bounded by $0 \le x \le 1$ and $0 \le y \le 1 - x$. Then

$$\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^{1-x} \left(\quad \right)$$

$$= \int_0 \int_0 (y - 0)\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{1}{2} (1-x)^2\, dx$$

$$= -\frac{1}{6} (1-x)^3 \Big|_{x=1}^{x=1} = \frac{1}{6}.$$

Exercise 1: Verify this answer using Strategy 1.

Here's a slick trick: let $F = \langle P, Q \rangle$ be any vector field with

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

Then for any simply connected D with boundary curve C (positively oriented),

$$\text{area}(D) = \iint_D 1 \, dA = \int_C F \cdot dr.$$

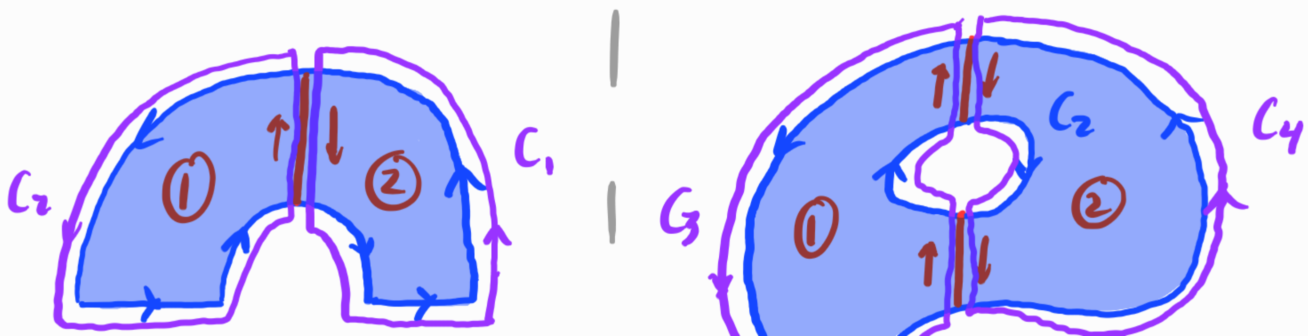
Exercise 2: Use this to find the area of the ellipse

$$ax^2 + by^2 = r^2.$$

Then verify your answer by computing the double integral using polar coordinates.

Hint: The ellipse can be parametrized by $r(t) = \langle \frac{r}{\sqrt{a}} \cos t, \frac{r}{\sqrt{b}} \sin t \rangle$ for $0 \le t \le 2\pi$.

Note: Green's Theorem also applies to non-simply connected regions:



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= (1) + (2)$$

$$= \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$$

$$= \int_C F \cdot dr$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= (1) + (2)$$

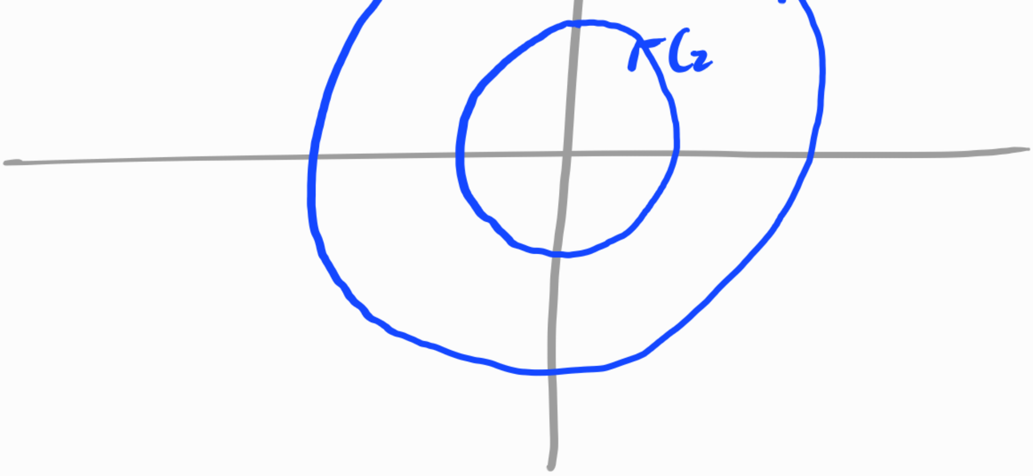
$$= \int_{C_3} F \cdot dr + \int_{C_4} F \cdot dr$$

$$= \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr.$$

Exercise 3: Solve the following integrals.

(a) $\int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$, where

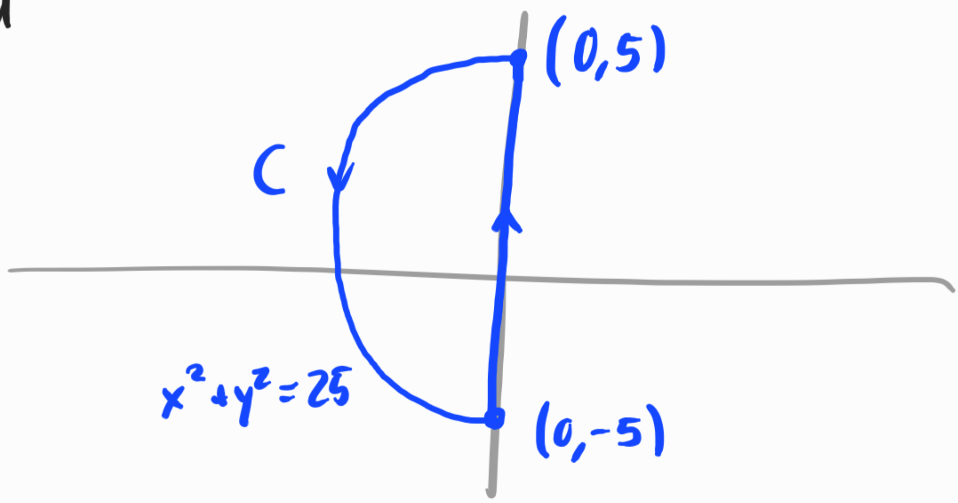
$F = \langle y^3, -x^3 \rangle$ and



$C_1 : x^2+y^2=4 \qquad C_2 : x^2+y^2=1$

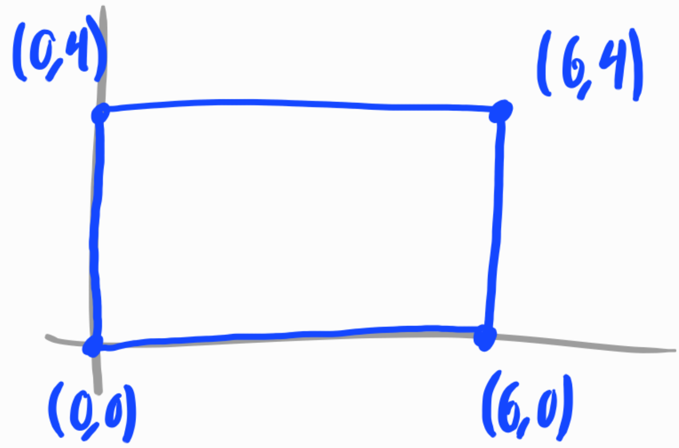
(b) $\int_C F \cdot dr$ where $F = \langle yx^2, -x^2 \rangle$

and



(c) $\int_C F \cdot dr$ where $F = \langle y^4 - 2y, 6x - 4xy^3 \rangle$

and



Curl and Divergence

The expression $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is a special case of a more general operation on a vector field.

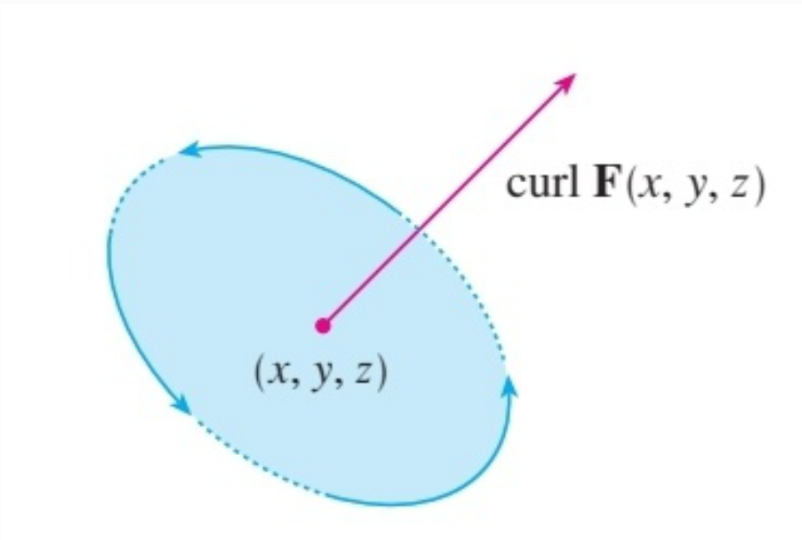
Def The curl of a vector field

$F = \langle P, Q, R \rangle$ in $\mathbb{R}^3$ is the vector field

$$\text{curl}(F) = \nabla \times F$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & P & i \\ \frac{\partial}{\partial y} & Q & j \\ \frac{\partial}{\partial z} & R & k \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle .$$



Note: If $F = \langle P, Q, 0 \rangle$ and P and Q only depend on x and y,

$$\mathrm{curl}(F) = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

Theorem If F is conservative,

$$\mathrm{curl}(F) = 0.$$

the vector field $\langle 0, 0, 0 \rangle$

Conversely:

Theorem If F is a vector field in $\mathbb{R}^3$ whose partial derivatives are continuous and $\operatorname{curl}(F) = 0$, F is conservative.

Another important operation on vector fields in $\mathbb{R}^3$ is:

$\boxed{\text{Def}}$ The **divergence** of $F = \langle P, Q, R \rangle$ is the scalar function

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$\boxed{\text{Theorem}}$ If P, Q and R have continuous second order partial derivatives, then

$$\mathrm{div}(\mathrm{curl}(F)) = 0.$$

Exercise 4: (a) Determine whether

$$F = \langle x^2 y,\ xyz,\ -x^2 y^2 \rangle$$

is a conservative vector field or not.

If it is, find $f(x,y,z)$ with $\nabla f = F$.

(b) Compute $\mathrm{div}(F)$ for the same F.

(c) Verify that $\mathrm{div}(\mathrm{curl}(F)) = 0$ if

your answer to (a) was no.

Exercise 5: Find div(F) and curl(F) for $F = \langle x^2y,\ 3x - z^3,\ 4y^2 \rangle$.

Surface Area, Revisited

We defined a curve C to be a vector-valued function with one input,

$$r(t) = \langle x_1(t), \ldots, x_n(t) \rangle.$$

The length of C from $t = a$ to $t = b$

was computed by

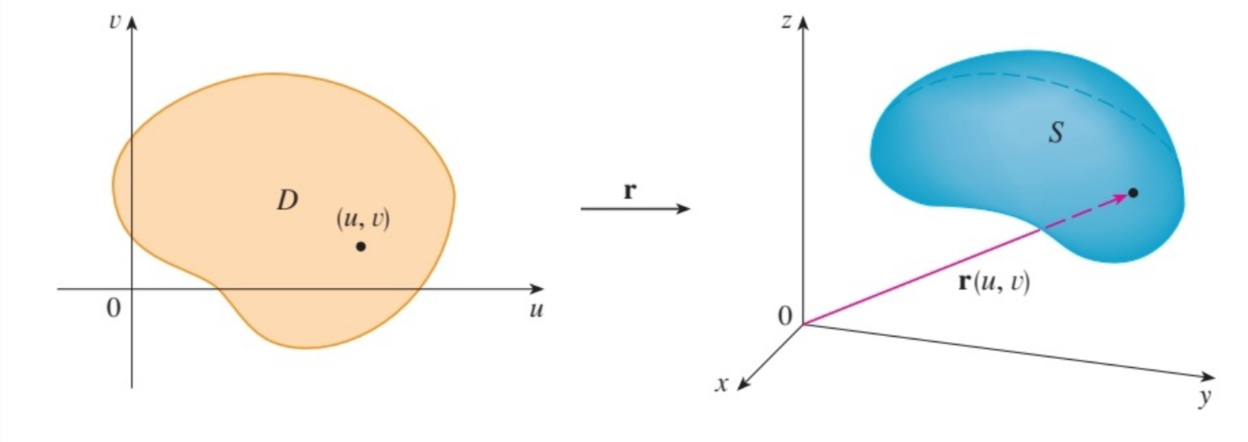
$$\int_C 1\, ds = \int_a^b |r'(t)|\, dt.$$

A vector-valued function of the form

$$r(s,t) = \langle x_1(s,t), \dots, x_n(s,t) \rangle$$

is called a parametric surface.

As (s,t) varies throughout a region D in the st-plane, $r(s,t)$ traces out a segment of a surface in $\mathbb{R}^n$:



The tangent plane to a surface

$S: \ r(s,t) = \langle x(s,t), y(s,t) \rangle, \ (s,t) \text{ in } D$

at a point $(s,t) = (a,b)$ is given by

the normal equation

$$(r_s \times r_t) \cdot \vec{x} = 0$$

where $r_s = \frac{\partial r}{\partial s}(a,b)$ and $r_t = \frac{\partial r}{\partial t}(a,b)$.

Using the fact that the area of the patch of the tangent plane spanned by the two vectors r_s and r_t is given by $|r_s \times r_t|$, it makes sense to define surface area as follows.

$\boxed{\text{Def}}$ The **surface area** of a surface

$$S: r(s,t), \quad (s,t) \text{ in } D$$

is computed by

$$A(S) = \iint_D |r_s \times r_t| \, dA.$$

$\boxed{\text{Ex}}$ When S is a graph, $z = f(x,y)$,

we can parametrize it by

$$r(s,t) = \langle s, t, f(s,t) \rangle.$$

Then over a region D in $\mathbb{R}^2$,

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \; dA$$

as we learned earlier.

Exercise 6: Verify this by computing

$$|r_s \times r_t| = |r_x \times r_y|.$$

Next time: surface integrals.