Lecture 16.7

Last time:

- For a vector field $F = \langle P, Q, R \rangle$, its curl

$$\text{curl}(F) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

detects if F is conservative:

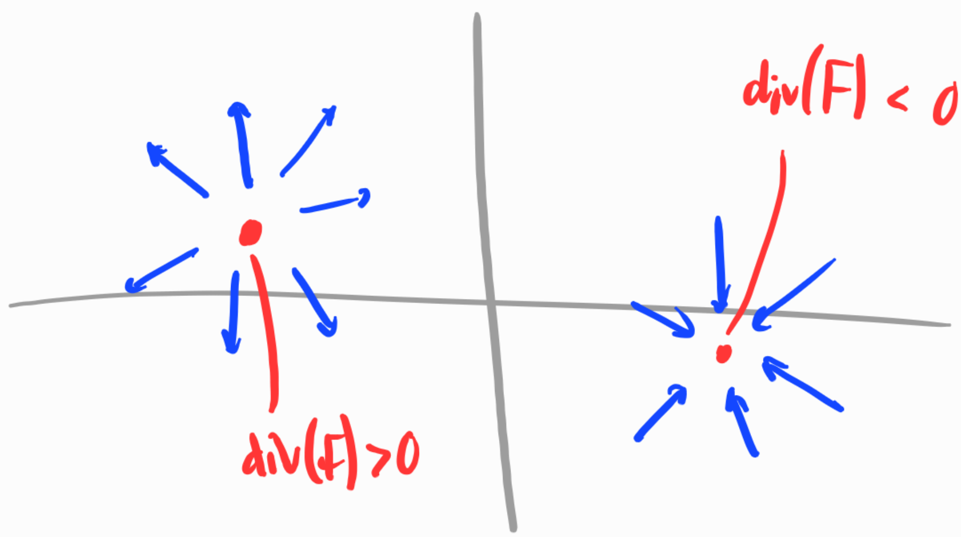
$$F = \nabla f \iff \text{curl}(F) = 0.$$

- The divergence of F,

$$\text{div}(F) = P_x + Q_y + R_z,$$

detects "sinks" and "[illegible]" F

detects sinks and sources in F:



- $\text{div}(\text{curl}(F)) = 0.$

Surface Area, Revisited

We defined a curve C to be a vector-valued function with one input,

$$r(t) = \langle x_1(t), \ldots, x_n(t) \rangle.$$

The length of C from $t = a$ to $t = b$

was computed by

$$\int_C 1 \, ds = \int_a^b |r'(t)| \, dt.$$

A vector-valued function of the form

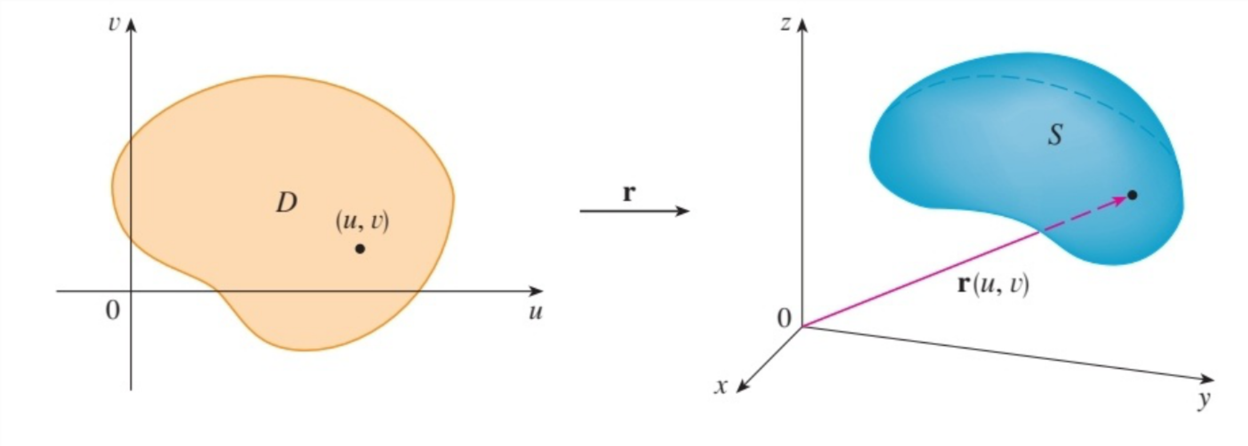
$$r(s,t) = \langle x_1(s,t), \ldots, x_n(s,t) \rangle$$

is called a parametric surface.

As (s,t) varies throughout a region

D in the st-plane, $r(s,t)$ traces

out a segment of a surface in $\mathbb{R}^n$:



The tangent plane to a surface

$$S: \ r(s,t) = \langle x(s,t), y(s,t) \rangle, \quad (s,t) \text{ in } D$$

at a point $(s,t) = (a,b)$ is given by

the normal equation

$$(r_s \times r_t) \cdot \vec{x} = 0$$

where $r_s = \frac{\partial r}{}(a,b)$

where $r_s = \frac{\partial r}{\partial s}(a,b)$ and $r_t = \frac{\partial r}{\partial t}(a,b)$.

Using the fact that the area of the patch of the tangent plane spanned by the two vectors r_s and r_t is given by $|r_s \times r_t|$, it makes sense to define surface area as follows.

$\boxed{\text{Def}}$ The **surface area** of a surface

$$S: r(s,t), \quad (s,t) \text{ in } D$$

is computed by

$$A(S) = \iint_D \underbrace{|r_s \times r_t| \, dA}_{dS}.$$

$\boxed{\text{Ex}}$ When S is a graph, $z = f(x,y)$,

we can parametrize it by

$$r(s,t) = \langle s, t, f(s,t) \rangle.$$

Then over a region D in $\mathbb{R}^2$,

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \; dA$$

as we learned earlier.

Exercise 1: Verify this by computing

$$|r_s \times r_t| = |r_x \times r_y|.$$

Surface Integrals

As we did with path integrals, let's now replace 1 with a function $f(x,y,z)$.

$\boxed{\text{Def}}$ The **surface integral** of a function $f(x,y,z)$ over a surface $S: r(s,t)$, (s,t) in D, is

$$\iint_S f\,dS = \iint_D f(r(s,t))\,|r_s \times r_t|\,dA.$$

$\boxed{Ex}$ Let's compute $\iint_S x^2\,dS$ where

$S : x^2 + y^2 + z^2 = 1.$

Parametrically, S can be described by

$$r(s,t) = \langle \cos(s)\sin(t), \sin(s)\sin(t), \cos(t) \rangle$$

with $0 \le s \le 2\pi$, $0 \le \phi \le \pi$.

think: $s = \theta$, $t = \phi$

Then

$$r_s = \langle -\sin(s)\sin(t), \cos(s)\sin(t), 0 \rangle$$

$$r_t = \langle \cos(s)\cos(t),\ \sin(s)\cos(t),\ -\sin(t) \rangle$$

$$r_s \times r_t = \langle -\cos(s)\sin^2(t),\ -\sin(s)\sin^2(t),$$

$$\underbrace{-\sin^2(s)\sin(t)\cos(t) - \cos^2(s)\sin(t)\cos(t)}_{-\sin(t)\cos(t)} \rangle$$

$$dS = |r_s \times r_t| \, dA$$

$$= \sqrt{\cos^2(s)\sin^4(t) + \sin^2(s)\sin^4(t) + \sin^2(t)\cos^2(t)} \, dA$$

$$= \sqrt{\sin^4(t) + \sin^2(t)\cos^2(t)} \, dA$$

$$= \sqrt{\sin^2(t)} \, dA = \sin(t) \, dA.$$

Then

$$\iint_S x^2\,dS = \int_0^{2\pi}\int_0^{\pi} (\cos(s)\sin(t))^2 \sin(t)\,dt\,ds$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2(s)\sin^3(t)\,dt\,ds$$

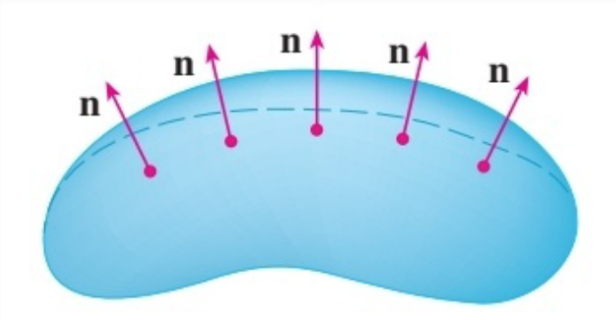
$$= \frac{4\pi}{3}.$$ (Check it yourself!)

Interpretation: $\iint_S f\,dS$ computes the mass of S with density function f.

$\boxed{\text{Def}}$ A surface S is **closed** if it is the boundary of some region R in $\mathbb{R}^3$. S is **positively oriented** if its unit normal vectors

$$n(s,t) = \frac{r_s \times r_t}{|r_s \times r_t|}$$

point <u>outward</u>, i.e. away from R.



outward pointing normal vectors

$\boxed{\text{Def}}$ For a vector field F in $\mathbb{R}^3$, the **surface integral** of F over a surface S: $r(s,t)$, (s,t) in D, or **flux** of F through S is

F through S, is

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS$$

$$= \iint_D F(r(s,t)) \cdot (r_s \times r_t) \, dA.$$

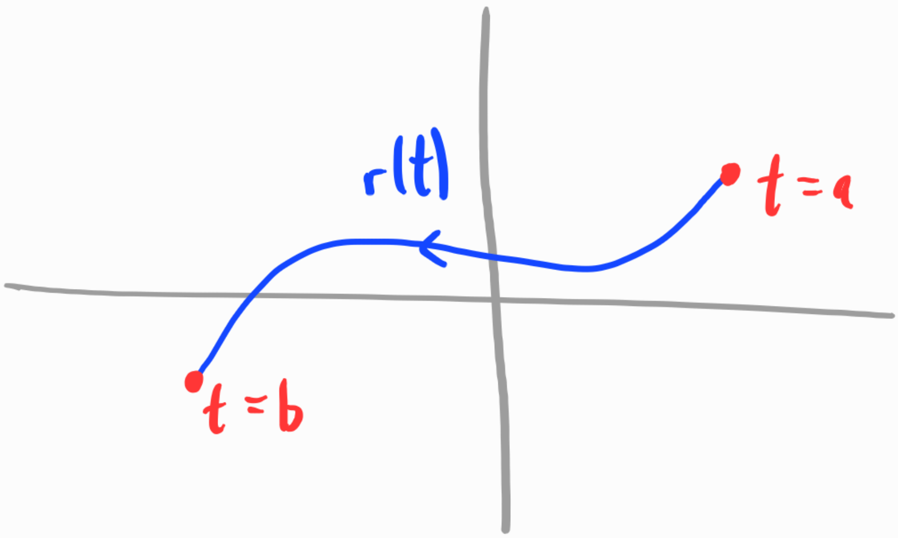
Interpretation: if F describes the flow rate of a liquid passing through S, then $\iint_S F \cdot dS$ calculates the total flow of liquid through S.

Exercise 2: Find $\iint_S F \cdot dS$ where

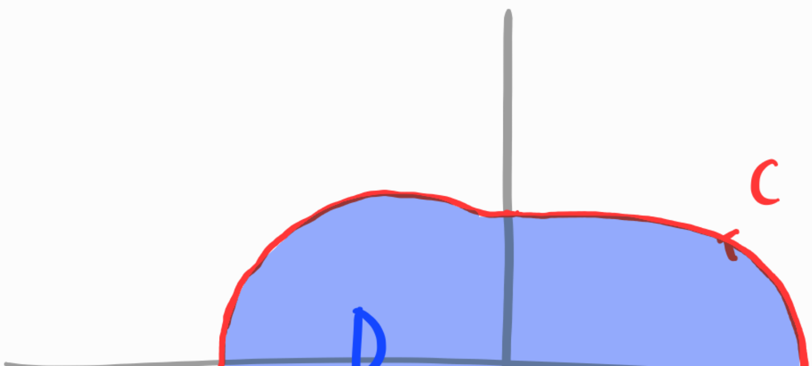
$F = \langle x, y, z^4 \rangle$ and $S: x^2 + y^2 + z^2 = 1.$

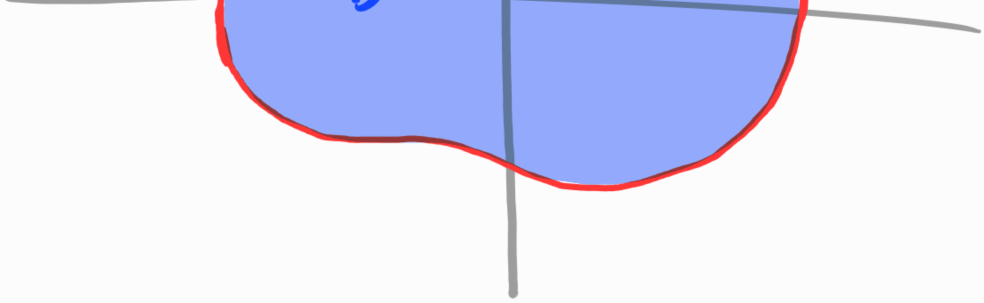
Stokes' and Divergence Theorems

We've seen that



$$\int_C F \cdot dr = f(r(b)) - f(r(a)) \quad \text{if } F = \nabla f$$



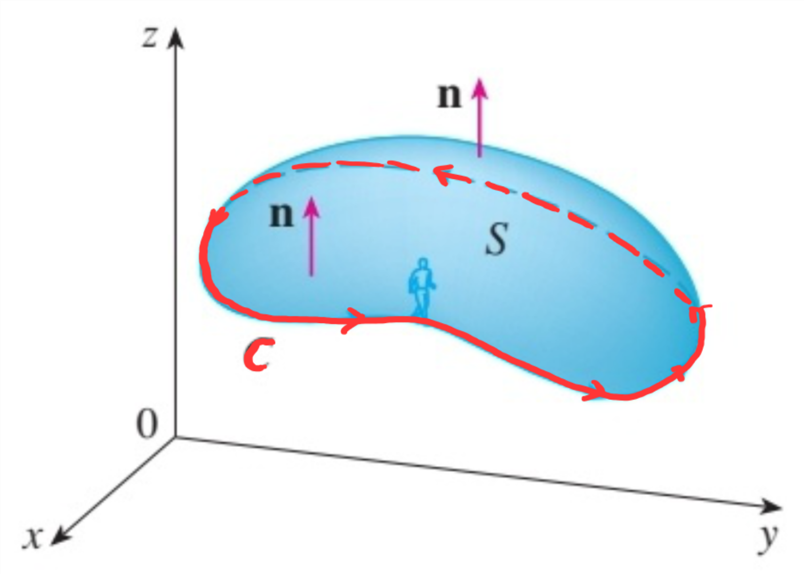


$$\iint_D G \, dA = \int_C F \cdot dr \quad \text{if} \quad G = \nabla \times F$$

Continuing this pattern, we have:

$\boxed{\text{Theorem (Stokes)}}$ Let S be a smooth surface, positively oriented, whose boundary is a simple, closed, smooth, positively oriented curve C. For any vector field F with continuous partial derivatives over S,

$$\int_C F \cdot dr = \iint_S \operatorname{curl}(F) \cdot dS.$$



That is,

$$\iint_S G \cdot dS = \int_C F \cdot dr \quad \text{if } G = \operatorname{curl}(F).$$

Theorem (Divergence) Let R be a region in $\mathbb{R}^3$ whose boundary is a positively oriented surface S. Then for any vector field F

with continuous partial derivatives on R,

$$\iint_S F \cdot dS = \iiint_R \operatorname{div}(F)\, dV.$$

That is,

$$\iiint_R f\, dV = \iint_S F \cdot dS \quad \text{if } f = \operatorname{div}(F).$$

General pattern:

$$\int_R dF = \int_{\partial R} F$$

where ∂R = boundary of R

dF = some sort of derivative of F

Exercise 3: Evaluate the following integrals.

(a) $\iint_S \text{curl}(F) \cdot dS$, $F = \langle z^2, -3xy, x^3y^3 \rangle$,

S: $z = 5 - x^2 - y^2$, $z \geq 1$, oriented upward.

(b) $\int_C F \cdot dr$, $F = \langle z^2, y^2, x \rangle$, C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, oriented counterclockwise.

(c) $\iint$ ...

$\iint_S F \cdot dS$, $F = \langle xy, \frac{-1}{2}y^2, z \rangle$, S is

made up of the three surfaces

$z = 4 - 3x^2 - 3y^2$, $1 \le z < 4$ on top,

$x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides,

$z = 0$ on bottom.

Strategies for surface integrals:

(1) If S is the boundary of a closed

region R, use the Divergence Theorem:

$$\iint_S F \cdot dS = \iiint_R \operatorname{div}(F)\, dV.$$

(2) Otherwise, find the boundary $C = \partial S$ and use Stokes' Theorem:

$$\iint_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr.$$