In Lecture 5.1, we proved:

Lemma Let p and q be distinct primes.

(a) $\phi(pq) = (p-1)(q-1)$.

(b) For any $k \geq 1$, $\phi(p^k) = p^{k-1}(p-1)$.

This allows us to prove:

Theorem If $n \in \mathbb{N}$ has prime factor-ization $n = p_1^{k_1} \cdots p_r^{k_r}$, where the p_j are distinct primes and $k_j \in \mathbb{N}$, then

$$\phi(n) = p_1^{k_1-1}(p_1-1)\cdots p_r^{k_r-1}(p_r-1).$$

Pf: Our proof of part (a) of the Lemma extends to the following situation: if $\gcd(a,b)=1$ then

$$\phi(ab) = \phi(a)\phi(b),$$

Combining this with (b) gives the full formula. $\square$

Exercise 3: Write out the details of the above statement about

$\phi(ab)$ for yourself.

$\boxed{\text{Ex}}$ $1000 = 2^3 \cdot 5^3$ so the Theorem says

$$\phi(1000) = 2^2(2-1)\,5^2(5-1)$$
$$= 400.$$

So 400 of the first 1000 natural numbers do not share a factor with 1000.