Lecture 5.2

Last time:

- There are infinitely many primes.
- All odd primes fall into the congruence classes $1 \pmod 4$ and $3 \pmod 4$, but we don't yet know how they're distributed.

Theorem There are infinitely many primes $p \equiv 3 \pmod 4$.

Pf: Mimicking our proof that there

are infinitely many primes overall,

let's suppose $p_1, \dots, p_r$ are all the primes $\equiv 3 \pmod 4$, with $p_1 = 3$.

Multiplying them all together and adding 1 gives a new prime, we saw, but it could be $\equiv 1$ or $3 \pmod 4$.

Instead, consider

$$q = 4p_2 \cdots p_r + 3.$$

Since $q \in \mathbb{N}$, we can factor it as

$$q = q_1 \cdots q_s$$

for some primes q_j.

Note that at least one of the q_j must be $\equiv 3 \pmod 4$; otherwise

$$q_1 \cdots q_s \equiv 1 \cdots 1 \equiv 1 \pmod 4$$

whereas

$$q = 4p_2 \cdots p_r + 3 \equiv 3 \pmod 4.$$

Finally, none of the primes $3, p_2, \ldots, p_r$ divide q (why?) so q_j is a new prime $\equiv 3 \pmod 4$, a contradiction. $\square$

Exercise 1. What happens if

Exercise 1: What happens if you try a similar proof for primes $p \equiv 1 \pmod 4$? Could there still be infinitely many of these?

Exercise 2: Prove there are infinitely many primes $p \equiv 5 \pmod 6$.

Q: For what $a, n \in \mathbb{N}$ are there infinitely many primes $p \equiv a \pmod n$?

Stay tuned ...

Counting Primes

Q: How common are primes among all natural numbers?

More specifically, among the first N numbers, how many are prime?

Def The prime counting function is

$$\pi(x) = \#\{p \leq x \mid p \text{ is prime}\}.$$

can be any real number ≥ 1

Ex Here's a table for $x = N \in \mathbb{N}$

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N	$\pi(N)$
2	1
3	2
4	2
5	3
10	4
50	15
100	25
200	46
500	95
1000	168
5000	669

What's the pattern?

Here's another way of measuring prime growth.

For $x \geq 1$, look at the ratio

$$\frac{\pi(x)}{x} \approx \frac{\#\{p \leq x\}}{\#\{n \leq x\}}$$

$=$ if $x \in \mathbb{N}$

Let's add a column to our table:

N	$\pi(N)$	$\frac{\pi(N)}{N}$ (rounded)
2	1	0.5
3	2	0.33
4	2	0.5
5	3	0.6
10	4	0.4
50	15	0.3
100	25	0.25

200	46	0.23
500	95	0.19
1000	168	0.168
5000	669	0.134

See a pattern?

Of course it's easy to see:

$\boxed{\text{Prime Number Theorem}}$ The function $\pi(x)$ grows at approximately the same rate as the function $\frac{x}{\ln(x)}$. That is,

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln(x)} = 1$$

Just kidding about this being easy. It's actually a deep result in number theory that we won't prove this semester.

Corollary The prime density function $\frac{\pi(x)}{x}$ grows at approximately the same rate as $\frac{1}{\ln(x)}$.

Here's a table comparing some of these:

x	10	100	1000	10^4	10^6	10^9

$\pi(x)$	4	25	168	1229	78498	50847534
$x/\ln(x)$	4.34	21.71	144.76	1085.74	72382.41	48254942.43
$\pi(x)/(x/\ln(x))$	0.921	1.151	1.161	1.132	1.084	1.054

Let's instead look at the mod 4 prime counting functions:

$$\pi_{1,4}(x) = \{p \leq x \mid p \equiv 1 \pmod 4 \text{ is prime}\}$$

$$\pi_{3,4}(x) = \{p \leq x \mid p \equiv 3 \pmod 4 \text{ is prime}\}.$$

Here's a table (from "Prime Counting Races", Granville-Martin '06):

x	Number of primes $4n+3$ up to x	Number of primes $4n+1$ up to x
100	13	11
200	24	21
300	32	29
400	40	37
500	50	44

600	57	51
700	65	59
800	71	67
900	79	74
1000	87	80
2000	155	147
3000	218	211
4000	280	269
5000	339	329
6000	399	383
7000	457	442
8000	507	499
9000	562	554
10,000	619	609
20,000	1136	1125
50,000	2583	2549
100,000	4808	4783

It turns out (again, difficult to prove) that $\pi_{1,4}$ and $\pi_{3,4}$ grow at the same rate:

$$\lim_{x \to \infty} \frac{\pi_{1,4}(x)}{\pi_{3,4}(x)} = 1.$$

However it appears $\pi(x) > \pi(x)$

However, it appears $\pi_{3,4}(x) > \pi_{1,4}(x)$

for all x.

Crazy fact: $\pi_{1,4}(x)$ gains the lead infinitely often, e.g. for

$$x = 26861,\ 616841,\ \ldots$$

However, the following is still an open question, verified up to large values of x but lacking a formal proof:

Conjecture: $\pi_{3,4}(x) > \pi_{1,4}(x)$ most of the time. More specifically,

$$\lim_{x \to \infty} \frac{\#\{x \mid \pi_{3,4}(x) > \pi_{1,4}(x)\}}{x} = 1.$$

More open conjectures include:

$\boxed{\text{Twin Primes Conjecture}}$ There are infinitely many primes p such that $p+2$ is also prime.

$\boxed{\text{Goldbach Conjecture}}$ Every even number $n \geq 4$ is the sum of two primes, $n = p + q$.

$\boxed{\text{Conjecture}}$ There are infinitely many primes of the form $p = n^2 + 1$.

These are some of the most famous open problems in math, despite the simplicity of their statements.

Exercise 3: Verify the patterns suggested in each conjecture for all natural numbers / prime numbers up to 100.

Next time: midterm review.