Lecture 8.1

Last time:

- A point $(x,y) \in \mathbb{R}^2$ is constructible if it can be obtained from the following procedures, starting from $(0,0)$ and $(1,0)$:

(1) Construct a straight line through any pair of constructed points.

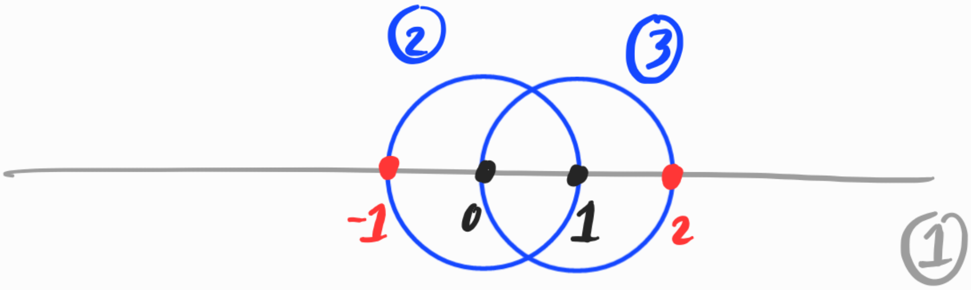
(2) Construct a circle centered at a constructed point passing through another constructed point.

(3) Any intersections become new constructed

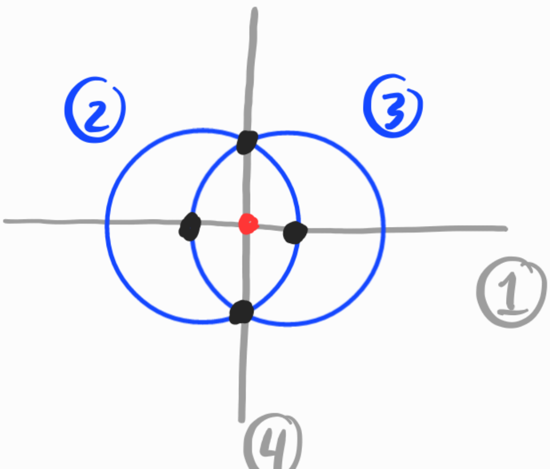
points.

Things we can construct

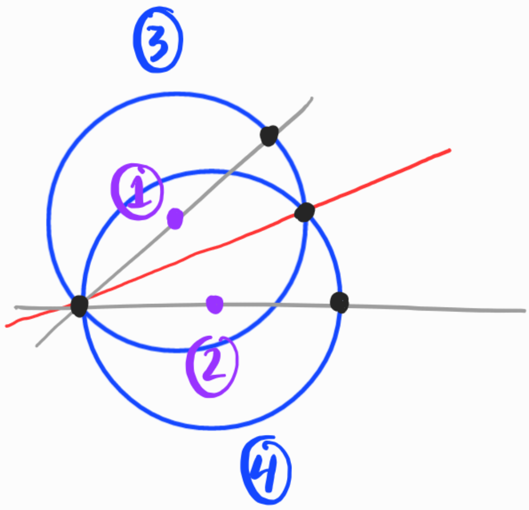
Integers :



Midpoints :



Angle bisection:



Other constructions:

- duplicating an angle
- perpendicular lines (see midpoints above)
- parallel lines
- similar triangles

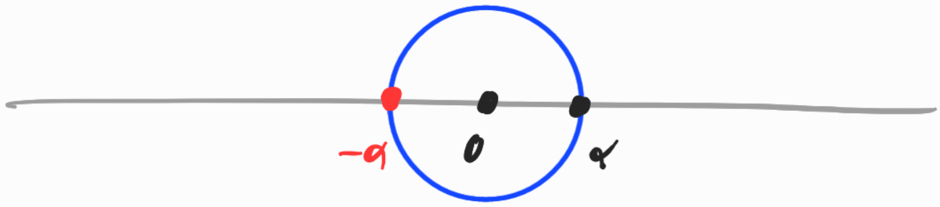
To make use of field theory, it's helpful

to change perspective and view coordinates

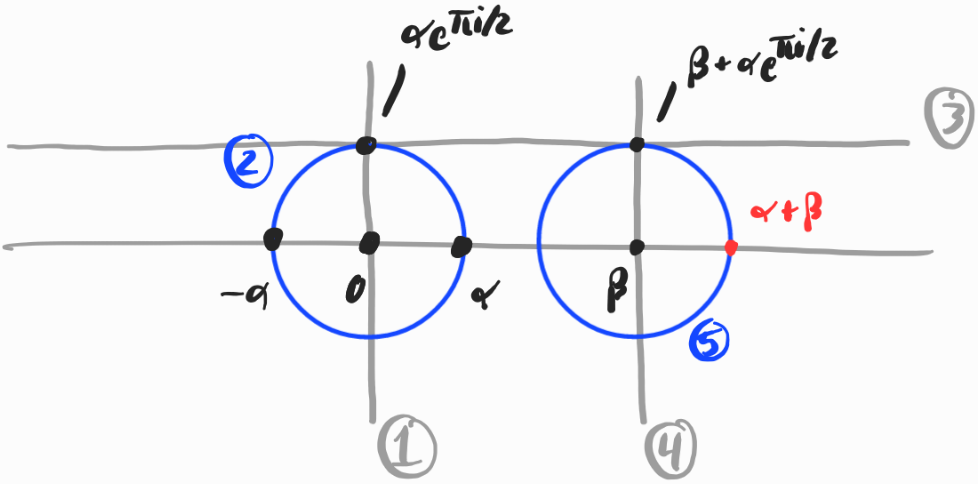
$(x,y) \in \mathbb{R}^2$ as complex numbers $x+iy \in \mathbb{C}$.

Theorem Let $C = \{\alpha \in \mathbb{C} \mid \alpha \text{ is constructible}\}$. Then C is a subfield of $\mathbb{C}$.

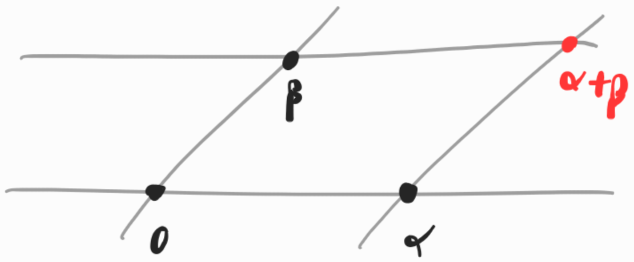
Pf: Let $\alpha, \beta \in C$. Then $-\alpha \in C$:



and if $0, \alpha, \beta$ are colinear, $\alpha + \beta \in C$:

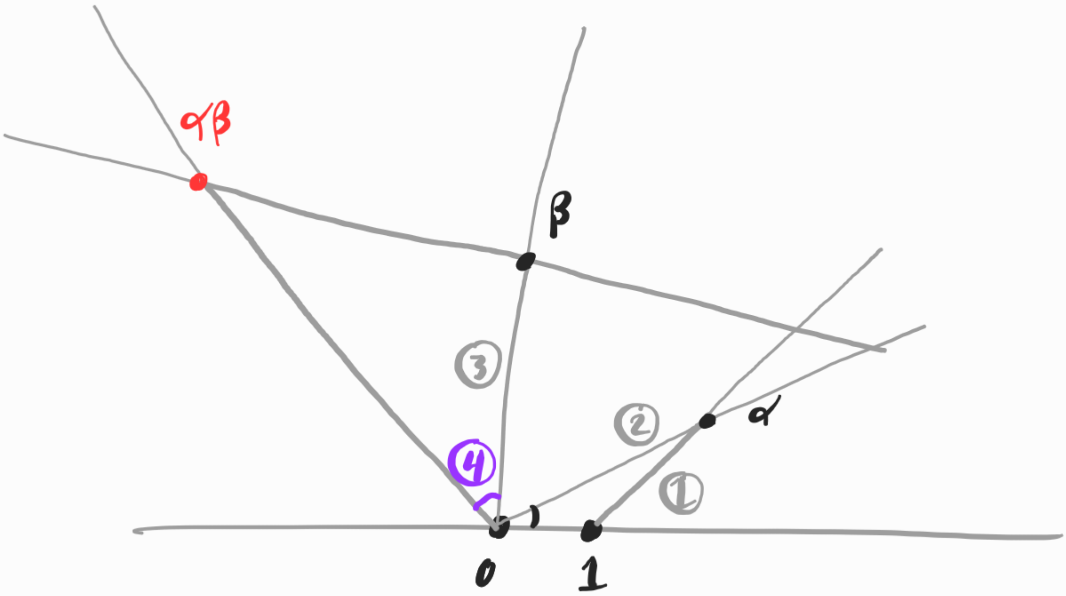


If $0, \alpha, \beta$ are not colinear, the parallelogram spanned by α, β constructs $\alpha + \beta$:



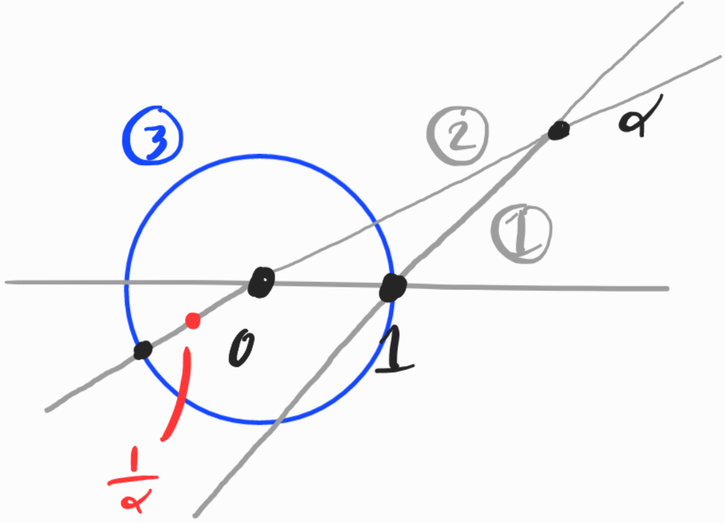
This shows C is an abelian subgroup of $\mathbb{C}$.

Next, $\alpha\beta \in C$:



(Check: $\Delta\, 0\beta(\alpha\beta)$ is similar to $\Delta\, 01\alpha$ and $\alpha\beta$ is the intersection point. Hint: look at the angles.)

Finally, if $\alpha \neq 0$, $\frac{1}{\alpha} \in \mathbb{C}$:



This shows C is a subfield of $\mathbb{C}$. $\square$

Theorem $\alpha \in C$ if and only if

$$\alpha \in \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$$

for some $d_1, \ldots, d_n \in \mathbb{C}$ with $d_1 \in \mathbb{Q}$ a nonsquare and $d_j \in \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_{j-1}})$ for each $2 \leq j \leq n$.

Pf: ($\Longrightarrow$) We will prove this by showing that for any subfield $K \subseteq \mathbb{C}$, the intersections of any lines and circles through points in K

lie in K or $K(\sqrt{\alpha})$ for some $\alpha \in K$.

Case 1: two lines are described by a linear system

$$ax + by = c$$
$$dx + ey = f$$

with $a, b, c, d, e, f \in K$. By linear algebra, the intersection point is also in K.

Case 2: a line and a circle intersect if

$$ax + by = c$$
$$(x-d)^2 + (y-e)^2 = f$$

has a solution. Solving for y and substituting yields

or x if $b=0$

$$Ax^2 + Bx + C = 0$$

where $A = 1 + \frac{a^2}{b^2}$

$B = -2d \ldots 2\frac{a}{\ldots}\left(\frac{c}{\ldots} \ldots\right) \in K$

$$\left.\begin{aligned} B &= 2d - 2b\left(\frac{c}{b} + e\right) \\ C &= d^2 + \left(\frac{c}{b} + e\right)^2 - f. \end{aligned}\right\} \in K$$

This has a solution in K if $B^2 - 4C$ has a square root in K; otherwise it has a solution in $K(\sqrt{\alpha})$ where $\alpha = B^2 - 4C$.

Case 3: similarly, two circles intersect in the roots of a quadratic equation.

Exercise 1: write out the details for Case 3.

($\Leftarrow$) We prove this by induction. First, $\mathbb{Q} \subseteq C$ by the previous Theorem.

Now assume $\mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_{n-1}}) \subseteq C$; we want to show that $\mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n}) = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_{n-1}})(\sqrt{d_n}) \subseteq C$

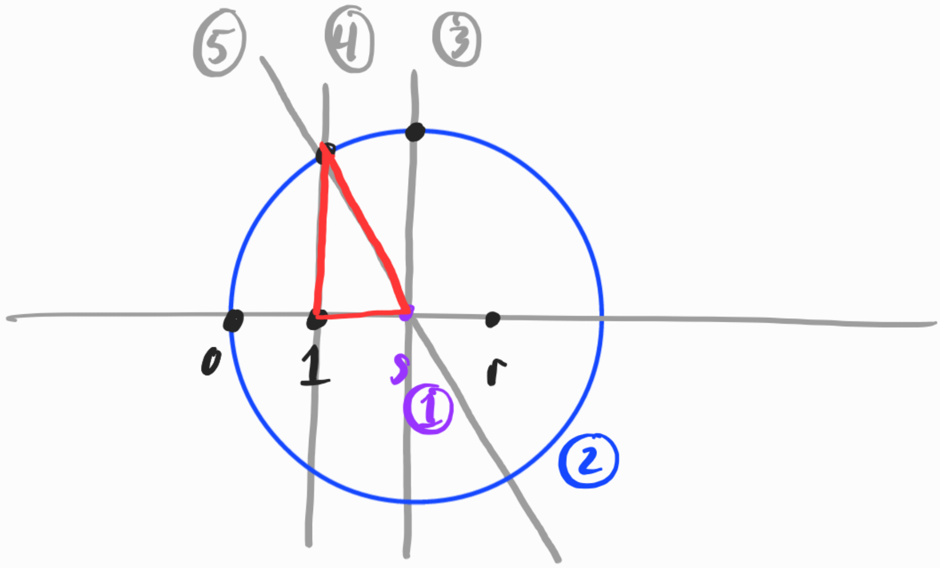
as well.

It's enough to show $\sqrt{d_n}$ is constructible since C is a field.

Write $d_n = re^{i\theta}$, so that $\sqrt{d_n} = \sqrt{r}\, e^{i\theta/2}$.

Since we can bisect any angle, it's enough to show $\sqrt{r} \in C$.

Here's a picture:



By construction, the legs of the red triangle are $s-1$ and

$$\sqrt{s^2-(s-1)^2} = \sqrt{2s-1} = \sqrt{r}.$$

Once we have a line segment of length $\sqrt{r}$ constructed, we can rotate it to construct $1+\sqrt{r} \in C$, so $\sqrt{r} = (1+\sqrt{r})-1 \in C$. $\square$

Corollary If $\alpha \in C$ then $[\mathbb{Q}(\alpha):\mathbb{Q}]$ is a power of 2.

Corollary It is impossible to square a circle.

Pf: Given a circle of radius r,

a square with the same area has sides of length $r\sqrt{\pi}$.

But π is transcendental so

$$[\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}] = \infty$$

which means $\sqrt{\pi} \notin C$. $\square$

Corollary It is impossible to double a cube.

Pf: Given a cube, say of unit volume, a cube of double the volume has sides of length $\sqrt[3]{2}$.

But the minimal polynomial of $\sqrt[3]{2}$ is

$x^3 - 2$ (check it!) so

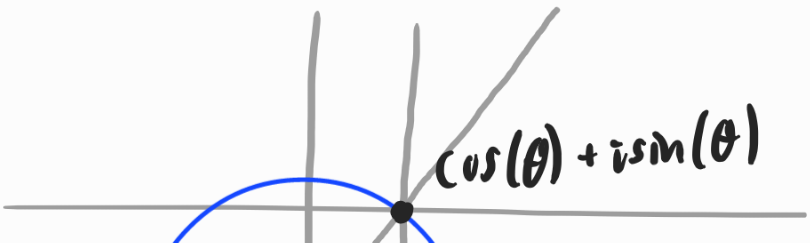
$$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$$

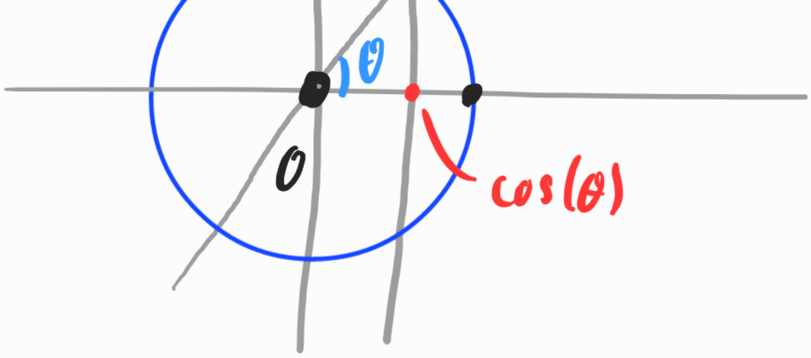
which is not a power of 2. $\square$

$\boxed{\text{Corollary}}$ It is impossible to trisect an angle.

Pf: Suppose θ has been constructed.

Then $\cos(\theta)$ is also constructible:





This actually allows to trisect some angles, e.g. $\pi/2$, but not all angles.

Let $\theta = \pi/9$, which would trisect the constructible angle $\pi/3$.

Then $2\cos(\theta) = e^{\pi i/9} + e^{-\pi i/9}$ which satisfies

$$(2\cos(\theta))^3 = \left(e^{\pi i/9} + e^{-\pi i/9}\right)^3$$

$$= e^{\pi i/3} + 3e^{\pi i/9} + 3e^{-\pi i/9} + e^{-\pi i/3}$$

$$= \underline{2\cos(\pi/3)} + 3(\underline{2\cos(\theta)}).$$

This shows that $\alpha = \cos(\theta)$ is a root of $8x^3 - 6x - 1$, which is irreducible (check it!) and therefore

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3.$$

Hence $\cos(\theta)$ is not constructible, so θ is not constructible. $\square$

Exercise 2: Prove that $2\pi/3$ cannot be trisected. Can you characterize those angles which can be trisected?

Next time: constructible polygons.