Modular Forms in Characteristic p

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Introduction

Goals:

(1) Provide a geometric construction of modular forms that works over any field.

(2) Understand classical dimension formulas as artifacts of geometry.

(3) (Work in progress) Compute analogues of these formulas in characteristic p.

Motivation

Example

(Eisenstein series) Let E_k be the weight k modular form defined by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Then for any prime p,

$$E_{p-1} \equiv 1 \pmod{p}$$
$$E_{(p-1)p^a} \equiv 1 \pmod{p^{a+1}} \quad \text{for any } a \geq 2$$

$\implies$ think of coefficients of E_k as p-adic functions (in the variable k).

Serre: use p-adic modular forms to study p-adic L-functions.

Space of "analytic" p-adic modular forms is not well-behaved...

Motivation

Instead: modular forms have a natural interpretation as sections of certain line bundles over a *moduli space* – hence the name ‘modular’ form perhaps?

For me (= algebraic geometer), moduli spaces are great:

- Rich in structure
- Often capture deeper information than the individual objects making up the ‘points’ of the space

Katz: construct modular forms geometrically using line bundles.

Motivation

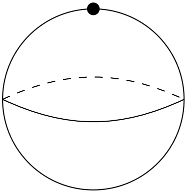
Example

$\Gamma = SL_2(\mathbb{Z})$ acts on $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ by fractional linear transformations.

$Y = \mathfrak{h}/\Gamma$ is an affine curve, projective closure $X \cong \mathbb{P}^1_{\mathbb{C}}$ *is the Riemann sphere*, and

modular forms of weight $2k \longleftrightarrow$ certain sections of line bundle $\omega_X^{\otimes k}$

where $f(z) \in \mathcal{M}_{2k} \leftrightarrow f(z)\,dz^k$.



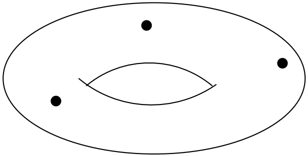
Riemann sphere
with cusp at ∞

Motivation

Example

More generally, for congruence subgroups $\Gamma \leq SL_2(\mathbb{Z})$, $Y(\Gamma) := \mathfrak{h}/\Gamma$ is an affine curve, projective closure $X(\Gamma)$ is a *Riemann surface*.

modular forms of weight $2k$ and level Γ $\longleftrightarrow$ certain sections of $\omega_{X(\Gamma)}^{\otimes k}$.



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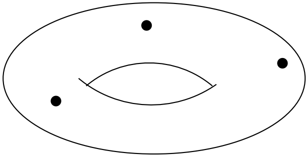

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Which sections correspond to modular forms?

Katz Modular Forms

Question

Can modular forms be defined over fields other than $\mathbb{C}$?

Another reason to ask this: modular curves $X(\Gamma)$ can be defined over number fields.

Arithmetic geometry: study them over $\mathbb{F}_p$ + local-global voodoo

Moduli Spaces of Elliptic Curves

Key point: $Y(\Gamma) =$ moduli space of elliptic curves with *level structure*.

Example

$Y_0(N) := Y(\Gamma_0(N))$ where

$$\Gamma_0(N) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N$$

parametrizes pairs (E, C) where $E =$ elliptic curve, $C \subseteq E(\mathbb{C})$ cyclic order N (up to iso.)

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$Y_1(N) := Y(\Gamma_1(N))$ where

$$\Gamma_1(N) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N$$

parametrizes pairs (E, P) where E = elliptic curve, $P \in E(\mathbb{C})$ point of order N (up to iso.)

Key point: $Y(\Gamma) =$ moduli space of elliptic curves with *level structure*.

Therefore, we can interpret a modular form $f : \mathfrak{h} \to \mathbb{C}$ as a Γ-invariant differential form on the space of elliptic curves.

For an elliptic curve E over an arbitrary field K, let $\omega_{E/K}$ be the pullback of the canonical line bundle $\Omega^1_{E/K}$ to $\operatorname{Spec} K$.

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Definition

A **Katz modular form** of weight k over K is a choice of section $f(E/A)$ of $\omega_{E/A}^{\otimes k/2}$ for every K-algebra A and elliptic curve E/A satisfying:

(1) $f(E/A)$ is constant on isomorphism class of E/A.

(2) (Naturality) f commutes with pullback along $A \to B$.

(3) (Holomorphic condition) The "q-expansion" $f(\text{Tate curve})$ has coefficients in $K \otimes \mathbb{Z}[[q]]$.

Cusp forms are modular forms with q-expansion coefficients in $K \otimes q\mathbb{Z}[[q]]$.

Pocket version:

- Pull back $\Omega^1_{E/K}$ along basepoint $O : \mathrm{Spec}\, K \hookrightarrow E$ to get $\omega_{E/K}$.
- Choose compatible $f(E/A) \in H^0(A, \omega_{E/A}^{\otimes k/2})$.
- Enforce holomorphic (and cusp) conditions with a geometric version of q-expansion principle.

Example

Over $K = \mathbb{C}$, any classical modular form $f : \mathfrak{h} \to \mathbb{C}$ can be recovered from the geometric construction:

$$f(\tau) \longleftrightarrow f(\tau)\, d\tau = f(E_\tau/\mathbb{C}[j]) \in H^0(E_\tau, \Omega_{E_\tau}^{\otimes k/2})$$

where:

$$E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \quad \text{as a complex torus}$$

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Still need to figure out which sections of $\Omega_X^{\otimes k/2}$ are modular forms.

Dimension Formulas

Let $\mathcal{M}_k$ be the vector space of modular forms of weight k and $X = X(SL_2(\mathbb{Z})) \cong \mathbb{P}^1_{\mathbb{C}}$.

Then we have a map

$$\mathcal{M}_k \longrightarrow H^0(X, \Omega_{X/K}^{\otimes k/2}), \quad f \longmapsto f\,\omega^{k/2} \quad (\omega \text{ “} = \text{” } dz)$$

Theorem

$\mathcal{M}_k =$
$\left\{\omega \in H^0(X, \Omega_{X/\mathbb{C}}^{\otimes k/2}) \,\middle|\, \mathrm{ord}_i(\omega) \geq -\frac{k}{2}, \mathrm{ord}_\rho(\omega) \geq -\frac{2k}{3}, \mathrm{ord}_\infty(\omega) \geq -k\right\}$

Proof.

Riemann-Roch for $X = \mathbb{P}^1_{\mathbb{C}}$. □

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Corollary (Valence Formula)

Let $k \in \mathbb{Z}$. Then

(1) *For $k < 0$, $k = 2$ and k odd, $\mathcal{M}_k = 0$.*

(2) *For $k \geq 0$ even,*

$$\dim_{\mathbb{C}} \mathcal{M}_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & k \not\equiv 2 \pmod{12}. \end{cases}$$

Dimension Formulas

More generally,

Theorem

For a congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$ and $k \geq 2$ even,

$$\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty$$

where $g = g(X(\Gamma))$ is the genus of the compactified modular curve and $\epsilon_2, \epsilon_3, \epsilon_\infty$ are the numbers of elliptic points of period $2, 3$ and cusps, respectively.

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They also follow from viewing $X(\Gamma)$ as a **complex orbifold**.

Dimension Formulas

Example

There is an isomorphism of orbifolds $X(SL_2(\mathbb{Z})) \cong \mathbb{P}(4,6)$, where $\mathbb{P}(4,6)$ denotes the **weighted projective line** with weights 4 and 6.

Under this isomorphism, $\mathcal{M}_k \cong H^0(\mathbb{P}(4,6), \mathcal{O}(k))$ and a basic combinatorial argument shows that

$$\dim H^0(\mathbb{P}(4,6), \mathcal{O}(k)) = \#\{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid 4x + 6y = k\}.$$

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This perspective is better:

- It recovers the valence formula
- The terms $\frac{k}{12}, \frac{k}{4}, \frac{k}{3}$, etc. can be interpreted as coefficients of *integral divisors* on the orbifold modular curve
- Also shows $\bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k \cong \mathbb{C}[x,y]$ where $\deg(x) = 4, \deg(y) = 6$
- Here, $x = E_4$ and $y = E_6$

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Theorem (Edixhoven)

For weights $k \geq 2$, levels $N \geq 1$ and primes $p \neq 2, 3$ with $p \nmid N$, Katz modular forms are just mod p reductions of classical modular forms.

But $\mathcal{M}_k(N; \mathbb{F}_p) \ncong \mathcal{M}_k(N) \otimes \mathbb{F}_p$ in general.

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Theorem (Katz-Mazur)

There is a modular form $A \in \mathcal{M}_{p-1}(1; \mathbb{F}_p)$ (the Hasse invariant) realizing the Frobenius action Frob_p^* on $H^1(E, \mathcal{O}_E) \cong H^0(E, \omega_E)^*$. Moreover, for $p \neq 2, 3$, $A \cong E_{p-1} \bmod p$ but for $p = 2, 3$, A is not the mod p reduction of any classical modular form.

A Possible Geometric Approach

- Fact: mod p analogues of elliptic curve level structure moduli problems are representable by schemes $X(N), X_0(N), X_1(N)$ over $\mathbb{F}_p$.
- Interpret $\mathcal{M}_k(N; \mathbb{F}_p)$ as sections of line bundles over $X_0(N)$.
- Apply Riemann–Roch to get analogues of dimension formulas.
- Even better, equip $X_0(N)$ with orbifold structure over $\mathbb{F}_p$ (stacky curve) and use stacky Riemann–Roch, stacky Riemann-Hurwitz, etc.
- Mazur: precise ramification structure of covers $X_1(N) \to X_0(N)$ is known (mod p).

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Idea:

- study tower of modular curves $X(p) \to X_1(p) \to X_0(p) \to X(1)$
- use techniques from [K. '19] when wild ramification shows up

Thank you!