Categorifying quadratic zeta functions

Andrew J. Kobin

OSU Number Theory Seminar

April 21, 2022

EMORY
UNIVERSITY

Joint work with Jon Aycock

Introduction
○●○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Introduction

Based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments: 23 pages
Subjects: **Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes: 11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as: arXiv:2011.13903 [math.NT]
                   (or arXiv:2011.13903v1 [math.NT] for this version)

and an upcoming preprint, tentatively titled "Categorifying quadratic zeta functions" (with Jon Aycock).

Introduction

Mysterious setup question: you probably know the definition of *the* zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(and you may know some examples of other zeta functions), but what is *a* zeta function?

Introduction
○○●○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Introduction

Mysterious setup question: you probably know the definition of *the* zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(and you may know some examples of other zeta functions), but what is *a* zeta function?

Introduction

Mysterious setup question: you probably know the definition of *the* zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(and you may know some examples of other zeta functions), but what is *a* zeta function?

Introduction
○○○○●○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

The Riemann Zeta Function

The **Riemann zeta function** is a meromorphic function $\zeta_{\mathbb{Q}}(s)$ on the complex plane defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It has:

- a product formula $\zeta_{\mathbb{Q}}(s) = \prod_p \dfrac{1}{1 - p^{-s}}$
- a functional equation $\xi(s) = \xi(1 - s)$ for the "completed zeta function" $\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta_{\mathbb{Q}}(s)$
- information about the distribution of the primes
- a Riemann hypothesis that predicts all "nontrivial" zeroes have $\mathrm{Re}(s) = \frac{1}{2}$.

Introduction
00000●000000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
00000

The Riemann Zeta Function

More generally, a **Dirichlet series** is a complex function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We will focus on the formal properties of Dirichlet series.

The coefficients $f(n)$ assemble into an **arithmetic function** $f : \mathbb{N} \to \mathbb{C}$. (Think: F is a generating function for f.)

Then $\zeta_{\mathbb{Q}}(s)$ is the Dirichlet series for $\zeta : n \mapsto 1$.

Introduction
000000●00000

Incidence Algebras
00000000000000000

Quadratic Zeta Functions
00000

Arithmetic Functions

The space of arithmetic functions $A = \{f : \mathbb{N} \to \mathbb{C}\}$ form an algebra under convolution:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\tfrac{n}{d}\right).$$

This identifies the algebra of formal Dirichlet series with A:

$$A \longleftrightarrow DS(\mathbb{Q})$$
$$f \longmapsto F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$
$$f * g \longmapsto F(s)G(s)$$
$$\zeta \longmapsto \zeta_{\mathbb{Q}}(s)$$

Introduction
00000000●0000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
00000

Number Fields

For a number field $K/\mathbb{Q}$, there is a zeta function $\zeta_K(s)$ defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}}{n^s}$$

where $I_K^+ = \{\text{ideals in } \mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

Like the Riemann zeta function, $\zeta_K(s)$ has:

- a product formula $\zeta_K(s) = \prod_{\mathfrak{p}} \dfrac{1}{1 - N(\mathfrak{p})^{-s}}$

- a functional equation

- information about the distribution of prime ideals

- a Riemann hypothesis

Introduction
00000000●000

Incidence Algebras
000000000000000000

Quadratic Zeta Functions
00000

Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits a map to $DS(\mathbb{Q})$:

$$N_* : A_K \longrightarrow A \cong DS(\mathbb{Q})$$

$$f \longmapsto \left(N_*f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right)$$

$$\zeta \longmapsto N_*\zeta \leftrightarrow \zeta_K(s)$$

Introduction
Incidence Algebras
Quadratic Zeta Functions
○○○○○○○○○●○○
○○○○○○○○○○○○○○○○○○
○○○○○

Varieties over Finite Fields

Let X be an algebraic variety over $\mathbb{F}_q$. Its point-counting zeta function is the power series

$$Z(X, t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

which has:

- a product formula $Z(X, t) = \displaystyle\prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}$

- a functional equation
- an expression as a *rational function*
- a Riemann hypothesis which is a theorem!

Introduction
0000000000000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
00000

Varieties over Finite Fields

Once again, we can formalize certain properties of $Z(X, t)$ in an algebra of arithmetic functions.

Let $Z_0^{\mathrm{eff}}(X)$ be the set of effective 0-cycles on X, i.e. formal $\mathbb{N}_0$-linear combinations of closed points of X, written $\alpha = \sum m_x x$. We say $\beta \leq \alpha$ if $\beta = \sum n_x x$ with $n_x \leq m_x$ for all $x \in |X|$.

Let $A_X = \{f : Z_0^{\mathrm{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

Introduction
0000000000●

Incidence Algebras
00000000000000000

Quadratic Zeta Functions
00000

Varieties over Finite Fields

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with
$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta)g(\alpha - \beta).$$

This time, there's no map to $DS(\mathbb{Q})$...

Introduction
○○○○○○○○○○●

Incidence Algebras
○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Varieties over Finite Fields

Let $A_X = \{f : Z_0^{\mathrm{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta)g(\alpha - \beta).$$

This time, there's no map to $DS(\mathbb{Q})$... but there's a map to the algebra of formal power series:

$$A_X \longrightarrow A_{\mathrm{Spec}\,\mathbb{F}_q} \cong \mathbb{C}[[t]]$$

$$f \leftrightarrow \sum_{n=0}^{\infty} f(n)t^n$$

$$f \longmapsto \text{``}\deg_*(f)\text{''}$$

$$\zeta \longmapsto \text{``}\deg_*(\zeta)\text{''} \leftrightarrow Z(X, t)$$

Introduction
000000000000

Incidence Algebras
●0000000000000000

Quadratic Zeta Functions
00000

What's really going on?

Introduction
000000000000

Incidence Algebras
●0000000000000000

Quadratic Zeta Functions
00000

What's really going on?

A, A_K and A_X are examples of the **reduced incidence algebra** of a poset.

Introduction
0000000000000

Incidence Algebras
0●0000000000000000

Quadratic Zeta Functions
00000

Incidence Algebra of a Poset

Let $(\mathcal{P}, \leq)$ be a poset and define $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$. Call $\mathcal{P}$ **locally finite** if every interval is finite.

Definition

The **incidence coalgebra** of a locally finite poset $\mathcal{P}$ is the free k-vector space $C(\mathcal{P})$ on the set of intervals in $\mathcal{P}$, with comultiplication

$$[x, y] \longmapsto \sum_{z \in [x,y]} [x, z] \otimes [z, y].$$

The **incidence algebra** of $\mathcal{P}$ is the dual $I(\mathcal{P}) = \mathrm{Hom}(C(\mathcal{P}), k)$ with multiplication

$$f \otimes g \longmapsto (f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z]) g([z, y]).$$

Introduction
00000000000

Incidence Algebras
0●0000000000000000

Quadratic Zeta Functions
00000

Incidence Algebra of a Poset

Let $(\mathcal{P}, \leq)$ be a poset and define $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$. Call $\mathcal{P}$ **locally finite** if every interval is finite.

Definition

The **incidence coalgebra** of a locally finite poset $\mathcal{P}$ is the free k-vector space $C(\mathcal{P})$ on the set of intervals in $\mathcal{P}$, with comultiplication

$$[x, y] \longmapsto \sum_{z \in [x,y]} [x, z] \otimes [z, y].$$

The **incidence algebra** of $\mathcal{P}$ is the dual $I(\mathcal{P}) = \mathrm{Hom}(C(\mathcal{P}), k)$ with multiplication

$$f \otimes g \longmapsto (f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z]) g([z, y]).$$

Think: elements in $I(\mathcal{P})$ are like arithmetic functions on the intervals in $\mathcal{P}$.

Introduction
0000000000000

Incidence Algebras
00●000000000000000

Quadratic Zeta Functions
00000

Reduced Incidence Algebra of a Poset

Definition

The **reduced incidence algebra** of $\mathcal{P}$ is the subalgebra $\widetilde{I}(\mathcal{P}) \subseteq I(\mathcal{P})$ of functions that are constant on isomorphism classes of intervals.

Think: elements in $\widetilde{I}(\mathcal{P})$ are like arithmetic functions on the isomorphism classes of intervals in $\mathcal{P}$.

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○●○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Reduced Incidence Algebra of a Poset

Definition

The **reduced incidence algebra** of $\mathcal{P}$ is the subalgebra $\widetilde{I}(\mathcal{P}) \subseteq I(\mathcal{P})$ of functions that are constant on isomorphism classes of intervals.

Think: elements in $\widetilde{I}(\mathcal{P})$ are like arithmetic functions on the isomorphism classes of intervals in $\mathcal{P}$.

Example

For the division poset $(\mathbb{N}, |)$, every interval is isomorphic to $[1, n]$ for some n. For $f \in \widetilde{I}(\mathbb{N}, |)$, write $f(n) := f([1, n])$. Then

$$\widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q})$$

$$f \longmapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Fact: the zeta function $\zeta : [x, y] \mapsto 1$ always lives in $\widetilde{I}(\mathcal{P})$.

Introduction
00000000000

Incidence Algebras
0000●0000000000000

Quadratic Zeta Functions
00000

Numerical Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○●○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} S_1 \overset{\longleftarrow}{\overset{\longrightarrow}{\underset{\longleftarrow}{\longrightarrow}}} S_2 \cdots .$$

Example

A poset $\mathcal{P}$ determines a simplicial set $N\mathcal{P}$ with:

- 0-simplices $=$ elements $x \in \mathcal{P}$
- 1-simplices $=$ intervals $[x, y]$
- 2-simplices $=$ decompositions $[x, y] = [x, z] \cup [z, y]$
- etc.

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○○●○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \rightleftarrows S_2 \cdots .$$

Example

More generally, any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

Introduction
○○○○○○○○○○○○

Incidence Algebras
○○○○○○●○○○○○○○○○○○

Quadratic Zeta Functions
○○○○○

Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence coalgebra** of a decomposition set S is the free k-vector space $C(S) = \bigoplus_{x \in S_1} kx$ with comultiplication

$$C(S) \longrightarrow C(S) \otimes C(S)$$
$$x \longmapsto \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma.$$

Introduction
000000000000

Incidence Algebras
0000000●000000000

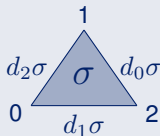Quadratic Zeta Functions
00000

Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication

$$I(S) \otimes I(S) \longrightarrow I(S)$$
$$f \otimes g \longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma) g(d_0\sigma).$$
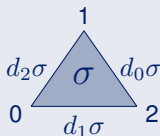
Introduction
000000000000

Incidence Algebras
000000000●000000000

Quadratic Zeta Functions
00000

Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the
zeta function $\zeta : x \mapsto 1$.

Introduction
0000000000

Incidence Algebras
0000000000000000

Quadratic Zeta Functions
00000

Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

Key takeaways:

(1) A zeta function is $\zeta \in I(S)$ for some decomposition set S.

(2) Familiar zeta functions like $\zeta_K(s)$ and $Z(X, t)$ are constructed from some $\zeta \in \widetilde{I}(S)$ by pushing forward to another reduced incidence algebra which can be interpreted in terms of generating functions:

$$\text{e.g.} \quad \widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q}), \qquad \text{e.g.} \quad \widetilde{I}(\mathbb{N}_0, \leq) \cong k[[t]].$$

(3) Some properties of zeta functions can be proven in the incidence algebra directly:

$$\text{e.g.} \quad \zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1 - p^{-s}} \longleftrightarrow \widetilde{I}(\mathbb{N}, |) \cong \bigotimes_p \widetilde{I}(\{p^k\}, |).$$

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○●○○○○○○○

Quadratic Zeta Functions
○○○○○

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** ("linear algebra with sets"):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $\begin{array}{ccc} & M & \\ {}^{s}\swarrow & & \searrow^{t} \\ B & & C \end{array}$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

Introduction
○○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○●○○○○○○○○

Quadratic Zeta Functions
○○○○○

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** ("linear algebra with sets"):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $\begin{array}{ccc} & M & \\ {}^{s}\swarrow & & \searrow{}^{t} \\ B & & C \end{array}$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Introduction
0000000000000

Incidence Algebras
00000000000●0000000

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$

Introduction
00000000000

Incidence Algebras
000000000000●000000

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S) =$ free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$

Introduction
00000000000

Incidence Algebras
0000000000000●0000

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S) =$ free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$
dual space $I(S) = \mathrm{Hom}(C(S), k)$	dual space $I(S) := \mathrm{Lin}(\mathrm{Set}_{/S_1}, \mathrm{Set})$

Introduction
0000000000000

Incidence Algebras
00000000000000●000

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S) =$ free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$
dual space $I(S) = \mathrm{Hom}(C(S), k)$	dual space $I(S) := \mathrm{Lin}(\mathrm{Set}_{/S_1}, \mathrm{Set})$

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{array}{ccc} & M & \\ & {}^{s}\swarrow \quad \searrow^{t} & \\ S_1 & & * \end{array} \right)$$

Introduction
0000000000000

Incidence Algebras
000000000000000●00

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{array}{ccc} & M & \\ s\swarrow & & \searrow t \\ S_1 & & * \end{array} \right)$$

Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{array}{ccc} & S_1 & \\ \mathrm{id}\swarrow & & \searrow \\ S_1 & & * \end{array} \right)$$

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○●○

Quadratic Zeta Functions
○○○○○

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \left(\begin{array}{c} \nearrow M \searrow \\ {}^{s}\swarrow \quad \quad \searrow \\ S_1 \quad \quad \quad * \end{array} \right) \quad \text{and} \quad g = \left(\begin{array}{c} \nearrow N \searrow \\ {}^{t}\swarrow \quad \quad \searrow \\ S_1 \quad \quad \quad * \end{array} \right)$$

the convolution $f * g \in I(S)$ is represented by

$$f * g) = \left(\begin{array}{c} P \\ \swarrow \quad \searrow \\ S_2 \quad \quad M \times N \\ {}^{d_1}\swarrow \quad {}^{(d_2, d_0)}\searrow \quad {}^{s \times t}\swarrow \quad \searrow \\ S_1 \quad \quad S_1 \times S_1 \quad \quad * \end{array} \right)$$

Introduction
00000000000

Incidence Algebras
00000000000000000●

Quadratic Zeta Functions
00000

Abstract Incidence Algebras

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly

- General: most* zeta functions can be produced this way

- Functorial: to compare zeta functions, find the right map $S \to T$

- Structural: proofs are categorical, avoiding choosing elements
  (e.g. computing local factors of zeta functions explicitly is difficult)

- Fun! you can try it yourself

Introduction
○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
●○○○○

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

Introduction
00000000000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
●0000

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

In Aycock-K., we lift this formula to an equivalence of linear functors in $\widetilde{I}(\mathbb{Q}) := \widetilde{I}(\mathbb{N}, |)$:

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+$$

where $N : (I_K^+, |) \to (\mathbb{N}, |)$ is the norm and $\chi^+, \chi^- \in I(\mathbb{Q})$.

In the numerical incidence algebra, this becomes

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (\chi^+ - \chi^-) = \zeta_{\mathbb{Q}} * \chi.$$

Introduction
00000000000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
0●000

Other Quadratic Zeta Functions: Euler Characteristic

For a ramified double cover $f : Y \to X$ of Riemann surfaces, the
Riemann–Hurwitz formula says

$$\chi(Y) = 2\chi(X) + \sum_{y \in Y}(e_y - 1).$$

It should be possible to lift this to a formula in $I(SX)$, the incidence
algebra of the simplicial complex of X, guided by the fact that

$$(1 - t)^{-\chi(X)} = \sum_{n=0}^{\infty} \chi(X^{(n)})t^n$$

is a generating function for the Euler characteristics of (symmetric
powers of) X.

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○●○○

Other Quadratic Zeta Functions: Elliptic Curves

For an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E, t)$ can be written

$$Z(E, t) = \frac{1 - a_q t + q t^2}{(1 - t)(1 - q t)} = Z(\mathbb{P}^1, t) L(E, t).$$

We are working on lifting this formula to the (reduced) incidence algebra $\widetilde{I}(Z_0^{\mathrm{eff}}(\mathbb{P}^1))$. Pushing forward to $\widetilde{I}(Z_0^{\mathrm{eff}}(\operatorname{Spec} \mathbb{F}_q)) \cong k[[t]]$, it already reads

$$(\pi_E)_* \zeta_E = (\pi_{\mathbb{P}^1})_* \zeta_{\mathbb{P}^1} * L(E).$$

Introduction
00000000000

Incidence Algebras
0000000000000000000

Quadratic Zeta Functions
0000

More Dreams

These are some other things I want to do:

- Extend our approach to "higher degree covers", e.g. for any abelian number field $K/\mathbb{Q}$, $\zeta_K(s) = \prod_{\chi} L(\chi, s)$

- Study the zeta function of an algebraic stack $\mathcal{X} \to X$ in terms of ζ_X, e.g. over $\mathbb{F}_q$, Behrend defines $Z(\mathcal{X}, t)$ for such a stack.

- Construct the right incidence algebra to house the motivic zeta function $Z_{mot}(X, t) = \sum_{n=0}^{\infty} [X^{(n)}] t^n$. (See Das–Howe)

- Realize archimedean factors of completed zeta functions as elements of abstract incidence algebras, e.g. the factor at ∞ $\zeta_{\infty}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right)$.

Key insight: decomposition sets $\rightsquigarrow$ decomposition spaces

Introduction
○○○○○○○○○○○○○

Incidence Algebras
○○○○○○○○○○○○○○○○○○○

Quadratic Zeta Functions
○○○○●

Thank you!