Categorifying zeta and L-functions

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Joint work with Jon Aycock

Introduction

This talk is based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments:	23 pages; minor changes and additional references added
Subjects:	**Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes:	11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as:	arXiv:2011.13903 **[math.NT]**
	(or arXiv:2011.13903v2 **[math.NT]** for this version)

and

Categorifying quadratic zeta functions

Jon Aycock, Andrew Kobin

The Dedekind zeta function of a quadratic number field factors as a product of the Riemann zeta function and the L-function of a quadratic Dirichlet character. We categorify this formula using objective linear algebra in the abstract incidence algebra of the division poset.

Comments:	27 pages
Subjects:	**Number Theory (math.NT)**
MSC classes:	11M06, 11M41, 18N50, 06A11, 16T10
Cite as:	arXiv:2205.06298 **[math.NT]**
	(or arXiv:2205.06298v1 **[math.NT]** for this version)

and generalizations to L-functions (work in progress with J. Aycock)

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

motivic L-functions
$Z_{mot}(X, t)$

arithmetic L-functions
$\zeta_{\mathbb{Q}}(x), \zeta_K(s), L(E, s)$

local L-functions
$Z(X/\mathbb{F}_q, t), L(E/\mathbb{F}_q, t)$

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$$
\begin{tikzcd}
 & \begin{array}{c}\text{motivic } L\text{-functions}\\ Z_{mot}(X,t)\end{array} \arrow[dr, "\text{motivic measure } \#_q"] & \\
\begin{array}{c}\text{arithmetic } L\text{-functions}\\ \zeta_K(x), L(\chi,s), L(E,s)\end{array} & & \begin{array}{c}\text{local } L\text{-functions}\\ Z(X/\mathbb{F}_q,t), L(E/\mathbb{F}_q,t)\end{array} \arrow[ll, hook', "\text{local factors}"']
\end{tikzcd}
$$

Deep Fried Version

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Zeta Functions of Number Fields

Let $K/\mathbb{Q}$ be a number field. Its zeta function is the Dirichlet series

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s}$$

Has the Riemann Hypothesis been solved?

GPT-3 almost did it though...

Zeta Functions of Number Fields

We can formalize certain properties of $\zeta_K(s)$ in an algebra of "arithmetic functions".

Let I_K^+ be the set of ideals of $\mathcal{O}_K$, ordered by reverse inclusion $\mathfrak{a} \leq \mathfrak{b} \iff \mathfrak{b} \subseteq \mathfrak{a}$.

Let $A_K = \{f : I_K^+ \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{d} \leq \mathfrak{a}} f(\mathfrak{d}) g(\mathfrak{a}\mathfrak{d}^{-1}).$$

We call the distinguished element $\zeta : \mathfrak{a} \mapsto 1$ the *zeta function* of I_K^+.

Zeta Functions of Number Fields

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The norm map $N : I_K^+ \to I_{\mathbb{Q}}^+ \cong \mathbb{N}$ determines an algebra map

$$\begin{aligned} A_K &\longrightarrow A_{\mathbb{Q}} \cong DS(\mathbb{C}) \\ f &\leftrightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_*(\zeta) \leftrightarrow \zeta_K(s) \end{aligned}$$

What's really going on?

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These A_K are examples of **reduced incidence algebras**, which come from a much more general simplicial framework.

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Decomposition Sets

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \; \substack{\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow} \; S_2 \cdots .$$

Example

Any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

Decomposition Sets

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$\begin{tikzcd} S_0 \arrow[r, shift left] & S_1 \arrow[l, shift left] \arrow[l, shift left=2] \arrow[r] & S_2 \cdots . \end{tikzcd}$$

Example

I_K^+ determines a category with:

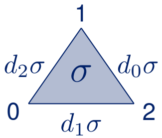
- 0-simplices $=$ objects $\mathfrak{a}$ in I_K^+
- 1-simplices $=$ relations $\mathfrak{b} \leq \mathfrak{a}$
- 2-simplices $=$ decompositions $\mathfrak{b} \leq \mathfrak{c} \leq \mathfrak{a}$
- etc.

Decomposition Sets

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Here, "arithmetic functions" on the 1-simplices $f : S_1 \to k$ are multiplied by

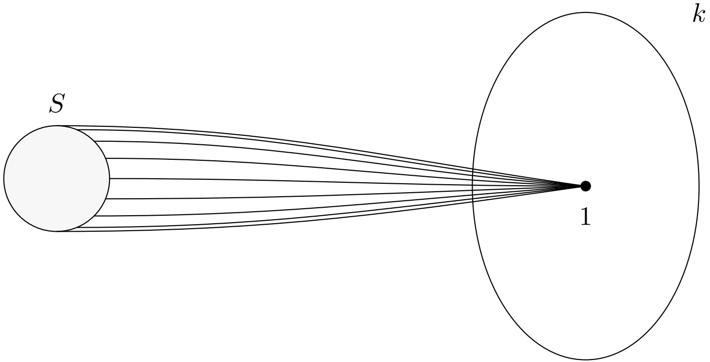
$$(f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma)g(d_0\sigma).$$



ζ

For a decomposition set S, let $I(S)$ be the algebra of "arithmetic functions" $I(S) = \mathrm{Hom}(k^{S_1}, k)$.

In $I(S)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.



Is everything zeta? ¯_(ツ)_/¯

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

Goal: connect different zeta and L-functions together using simplicial maps between their native incidence algebras

e.g. for $K/\mathbb{Q}$ quadratic (Aycock–K., 2022):

$$\begin{tikzcd} I(K) \arrow[dr] & & I(\chi) \arrow[dl] \\ & I(\mathbb{Q}) & \end{tikzcd}$$

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$$

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To bring L-functions into the mix, need **objective linear algebra**.

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** ("linear algebra with sets"):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B & & C \end{tikzcd}$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

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$\longleftarrow$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

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$I(S) = \mathrm{Hom}(k^{S_1}, k)$	$I(S) := \mathrm{Lin}(\mathrm{Set}_{/S_1}, \mathrm{Set})$

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ S_1 & & * \end{tikzcd} \right)$$

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Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{tikzcd} & S_1 \arrow[dl, "\text{id}"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \left(\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right) \quad \text{and} \quad g = \left(\begin{tikzcd} & N \arrow[dl, "t"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right)$$

the convolution $f * g \in I(S)$ is represented by

$$(f * g) = \left(\begin{tikzcd} & & P \arrow[dl] \arrow[dr] & & \\ & S_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & & M \times N \arrow[dl, "s \times t"] \arrow[dr] & \\ S_1 & & S_1 \times S_1 & & * \end{tikzcd} \right)$$

Quadratic Number Fields

For a quadratic extension $K/\mathbb{Q}$,

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where χ is the quadratic character cutting out $K/\mathbb{Q}$.

Theorem (Aycock–K., '22)

In the reduced incidence algebra $\widetilde{I}(\mathbb{Q}) := \widetilde{I}(\mathbb{N}, |)$, there is an equivalence of linear functors

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * L(\chi)^- \cong \zeta_{\mathbb{Q}} * L(\chi)^+$$

where $N : I_K^+ \to \mathbb{N}$ is the norm map and $L(\chi)^+$ and $L(\chi)^-$ are functors in $\widetilde{I}(\mathbb{N})$.

Taking cardinalities, the formula reads

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (L(\chi)^+ - L(\chi)^-) = \zeta_{\mathbb{Q}} * L(\chi).$$

Elliptic Curves

Analogously, for an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

Theorem (Aycock–K., '22+ϵ)

In the reduced incidence algebra $\widetilde{I}(E) := \widetilde{I}(Z_0^{\mathrm{eff}}(E))$, there is an equivalence of linear functors

$$\pi_* \zeta_E + \zeta_{\mathbb{P}^1} * L(E)^- \cong \zeta_{\mathbb{P}^1} * L(E)^+$$

where $\pi : E \to \mathbb{P}^1$ is a fixed double cover and $L(E)^+$ and $L(E)^-$ are functors in $\widetilde{I}(\mathbb{P}^1)$.

Pushing forward to $\widetilde{I}(\operatorname{Spec} \mathbb{F}_q)$ and taking cardinalities, it reads

$$\pi_* \zeta_E = \pi_* \zeta_{\mathbb{P}^1} * (L(E)^+ - L(E)^-) = \pi_* \zeta_{\mathbb{P}^1} * L(E).$$

Highlights and Dreams for the Future

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- It's pretty fun to prove things!

Future work:

- Motivic zeta functions $Z_{mot}(X, t)$
- Construct $\zeta_{\mathcal{X}}$ for an algebraic stack $\mathcal{X}$
- Lift L-functions $L(V)$ of Galois representations
- Archimedean zeta functions

Thank you!