Zeta functions and decomposition spaces

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POINT New Developments in Number Theory

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EMORY UNIVERSITY

Joint work with Bogdan Krstic, Jon Aycock

Introduction

Based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

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work in progress with B. Krstic and an upcoming preprint, tentatively titled “Categorifying quadratic zeta functions”, with J. Aycock.

Introduction

Mysterious setup question: you probably know the definition of *the* zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(and you may know some examples of other zeta functions), but what is *a* zeta function?

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The Riemann Zeta Function

The **Riemann zeta function** is a meromorphic function $\zeta_{\mathbb{Q}}(s)$ on the complex plane defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

More generally, a **Dirichlet series** is a complex function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We will focus on the formal properties of Dirichlet series.

The coefficients $f(n)$ assemble into an **arithmetic function** $f : \mathbb{N} \to \mathbb{C}$. (Think: F is a generating function for f.)

Then $\zeta_{\mathbb{Q}}(s)$ is the Dirichlet series for $\zeta : n \mapsto 1$.

Arithmetic Functions

The space of arithmetic functions $A = \{f : \mathbb{N} \to \mathbb{C}\}$ form an algebra under convolution:

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

This identifies the algebra of formal Dirichlet series with A:

$$\begin{aligned} A &\longleftrightarrow DS(\mathbb{Q}) \\ f &\longmapsto F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\ f * g &\longmapsto F(s)G(s) \\ \zeta &\longmapsto \zeta_{\mathbb{Q}}(s) \end{aligned}$$

Number Fields

For a number field $K/\mathbb{Q}$, there is a zeta function $\zeta_K(s)$ defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}}{n^s}$$

where $I_K^+ = \{\text{ideals in } \mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

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where $I_K^+ = \{$ideals in $\mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

Why it's a zeta function: $\zeta : \mathfrak{a} \mapsto \mathbf{1}$

Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits a map to $DS(\mathbb{Q})$:

$$\begin{aligned} N_* : A_K &\longrightarrow A \cong DS(\mathbb{Q}) \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_*\zeta \leftrightarrow \zeta_K(s) \end{aligned}$$

Varieties over Finite Fields

There's a similar story when X is an algebraic variety over $\mathbb{F}_q$, with point-counting zeta function

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

Why it's a zeta function: $Z(X,t) = \sum_{\alpha \in Z_0^{\mathrm{eff}}(X)} \mathbf{1} t^{\deg(\alpha)}$ where $Z_0^{\mathrm{eff}}(X) =$ effective 0-cycles on X. So $\zeta : \alpha \mapsto \mathbf{1}$.

There's an algebra A_X of functions on Z_0^{eff} which maps to the algebra of formal power series:

$$A_X \longrightarrow A_{\operatorname{Spec}\mathbb{F}_q} \cong \mathbb{C}[[t]]$$

$$f \leftrightarrow \sum_{n=0}^{\infty} f(n) t^n$$

$$f \longmapsto \text{“}\deg_*(f)\text{”}$$

$$\zeta \longmapsto \text{“}\deg_*(\zeta)\text{”} \leftrightarrow Z(X,t)$$

What’s really going on?

What's really going on?

A, A_K and A_X are examples of the **(reduced) incidence algebra** of a poset.

Incidence Algebra of a Poset

Let $(\mathcal{P}, \leq)$ be a poset and define $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$. Call $\mathcal{P}$ **locally finite** if every interval is finite.

Definition

The **incidence coalgebra** of a locally finite poset $\mathcal{P}$ is the free k-vector space $C(\mathcal{P})$ on the set of intervals in $\mathcal{P}$, with comultiplication

$$[x, y] \longmapsto \sum_{z \in [x,y]} [x, z] \otimes [z, y].$$

The **incidence algebra** of $\mathcal{P}$ is the dual $I(\mathcal{P}) = \mathrm{Hom}(C(\mathcal{P}), k)$ with multiplication

$$f \otimes g \longmapsto (f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z])g([z, y]).$$

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$$f \otimes g \longmapsto (f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z])g([z, y]).$$

Think: elements in $I(\mathcal{P})$ are like arithmetic functions on the intervals in $\mathcal{P}$.

Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftrightarrows S_1 \leftrightarrows S_2 \cdots .$$

Example

A poset $\mathcal{P}$ determines a simplicial set $N\mathcal{P}$ with:

- 0-simplices $=$ elements $x \in \mathcal{P}$
- 1-simplices $=$ intervals $[x, y]$
- 2-simplices $=$ decompositions $[x, y] = [x, z] \cup [z, y]$
- etc.

Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftleftarrows\!\!\!\rightarrow S_1 \cdots S_2 \cdots .$$

Example

More generally, any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

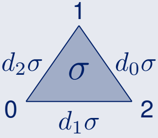
Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **incidence coalgebra** of a decomposition set S is the free k-vector space $C(S) = \bigoplus_{x \in S_1} kx$ with comultiplication

$$\begin{aligned} C(S) &\longrightarrow C(S) \otimes C(S) \\ x &\longmapsto \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma. \end{aligned}$$



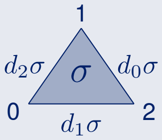
Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication

$$\begin{aligned} I(S) \otimes I(S) &\longrightarrow I(S) \\ f \otimes g &\longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma) g(d_0\sigma). \end{aligned}$$



Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

Key takeaways:

(1) A zeta function is $\zeta \in I(S)$ for some decomposition set S.

(2) Familiar zeta functions like $\zeta_K(s)$ and $Z(X, t)$ are constructed from some $\zeta \in \widetilde{I}(S)$ by pushing forward to another reduced* incidence algebra which can be interpreted in terms of generating functions:

$$\text{e.g.} \quad \widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q}), \qquad \text{e.g.} \quad \widetilde{I}(\mathbb{N}_0, \leq) \cong k[[t]].$$

(3) Some properties of zeta functions can be proven in the incidence algebra directly:

$$\text{e.g.} \quad \zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1 - p^{-s}} \longleftrightarrow \widetilde{I}(\mathbb{N}, |) \cong \bigotimes_p \widetilde{I}(\{p^k\}, |).$$

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

Theorem (Aycock–K.)

In $I(\mathbb{N}, |)$, there is an equivalence of functors

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+$$

where $N : (I_K^+, |) \to (\mathbb{N}, |)$ is the norm and $\chi^+, \chi^- \in I(\mathbb{N}, |)$.

In $A_{\mathbb{Q}} \cong DS(\mathbb{Q})$, this becomes

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (\chi^+ - \chi^-) = \zeta_{\mathbb{Q}} * \chi.$$

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** ("linear algebra in a category"):

Numerical	Objective
basis B	object B
vector v	morphism $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathcal{S}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathcal{S}_{/B} \to \mathcal{S}_{/C}$
tensor product $V \otimes W$	$\mathcal{S}_{/B} \otimes \mathcal{S}_{/C} \cong \mathcal{S}_{/B \times C}$

Abstract Incidence Algebras

How do we construct $I(S)$ as an “objective vector space”? Now, S can be any simplicial object in $\mathcal{S}$.

$$C(S) = \text{ slice category } \mathcal{S}_{/S_1} \qquad I(S) = \mathrm{Fun}(\mathcal{S}_{/S_1}, \mathcal{S}).$$

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathcal{S}_{/S_1} \to \mathcal{S}$ represented by a span

$$f = \left(\begin{array}{ccc} & M & \\ {}^{s}\swarrow & & \searrow^{t} \\ S_1 & & * \end{array} \right)$$

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{array}{ccc} & S_1 & \\ {}^{\mathrm{id}}\swarrow & & \searrow \\ S_1 & & * \end{array} \right)$$

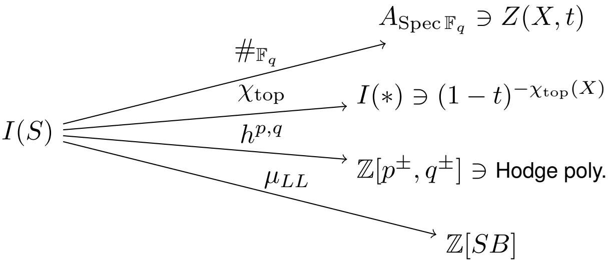
Motivic Zeta Function

Goal (joint with B. Krstic): lift the motivic zeta function of a k-variety

$$Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$$

to $\zeta \in I(S)$ for some decomposition space S.

More ambitious goal: represent motivic measures as maps between abstract incidence algebras, e.g.



Thank you!