Arithmetic Geometry and Stacky Curves

Andrew J. Kobin

ajkobin@emory.edu

Penn Algebra Seminar

March 22, 2024

EMORY UNIVERSITY

Introduction

Believe women.

Believe your colleagues.

Stop the cruelty.

Motivation: Find all integer solutions (x, y, z) to the generalized Fermat equation

$$Ax^p + By^q = Cz^r$$

for $A, B, C \in \mathbb{Z}$ and $p, q, r \geq 2$.

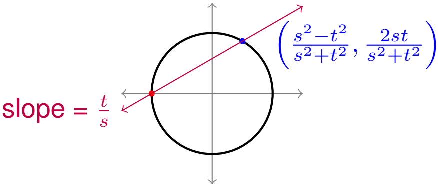
Generalized Fermat Equations

Motivation: Find integer solutions to $Ax^p + By^q = Cz^r$.

Example $((A,B,C) = (1,1,1), (p,q,r) = (2,2,2))$

Famously, there are infinitely many integer solutions to $x^2 + y^2 = z^2$, with primitive $(\gcd(x,y,z) = 1)$ solutions parametrized by

$$(x,y,z) = \left(\frac{s^2 - t^2}{2}, st, \frac{s^2 + t^2}{2}\right) \quad \text{for odd, coprime } s > t \geq 1.$$



Generalized Fermat Equations

Motivation: Find integer solutions to $Ax^p + By^q = Cz^r$.

Example $((A,B,C) = (1,1,1), (p,q,r) = (n,n,n))$

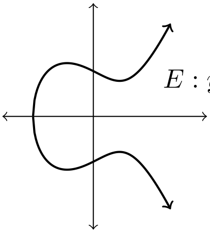
Also famously, there are *no* integer solutions to $x^n + y^n = z^n$ for $n > 2$.

Generalized Fermat Equations

Motivation: Find integer solutions to $Ax^p + By^q = Cz^r$.

Example $((A,B,C)=(1,1,1),(p,q,r)=(n,n,n))$

Also famously, there are *no* integer solutions to $x^n + y^n = z^n$ for $n > 2$. Assume n is prime. If (x_0, y_0, z_0) were such a solution, it would determine an elliptic curve



$$E : y^2 = x(x - x_0^n)(x + y_0^n)$$

Ribet showed E is not modular. However, Wiles showed all such elliptic curves are modular, a contradiction.

Generalized Fermat Equations

Takeaway: Integer solutions to $Ax^p + By^q = Cz^r$ can be studied using geometry.

Generalized Fermat Equations

Here are some more known cases of $Ax^p + By^q = Cz^r$.

- (Beukers, Darmon–Granville) Let $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$. The equation $x^p + y^q = z^r$ has infinitely many primitive solutions when $\chi > 0$ and finitely many when $\chi < 0$.

Generalized Fermat Equations

Here are some more known cases of $Ax^p + By^q = Cz^r$.

- (Beukers, Darmon–Granville) Let $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$. The equation $x^p + y^q = z^r$ has infinitely many primitive solutions when $\chi > 0$ and finitely many when $\chi < 0$.
- (Mordell, Zagier, Edwards) When $\chi > 0$, the primitive solutions to $x^p + y^q = z^r$ may always be parametrized explicitly (as in the $(2,2,2)$ case).
- (Fermat, Euler, et al.) The case $\chi = 0$ only occurs for $(2,3,6), (4,4,2), (3,3,3)$ and permutations of these. In each case, descent proves there are finitely many primitive solutions.
- $(2,3,7)$ was solved by Poonen–Schaeffer–Stoll (2007).
- $(2,3,8), (2,3,9)$ were solved by Bruin (1999, 2004).
- etc.

Generalized Fermat Equations

Question: How do we count solutions to such equations?

Generalized Fermat Equations

Question: How do we count solutions to such equations?

One strategy is to form the surface of (primitive, nontrivial) solutions in 3-dimensional space over $\mathbb{Z}$:

$$S = \operatorname{Spec}(\mathbb{Z}[x,y,z]/(Ax^p + By^q - Cz^r)) \smallsetminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}}.$$

For any ring R, this keeps track of the R-solutions:

$$S(R) = \{x, y, z \in R \mid Ax^p + By^q = Cz^r, \text{ nontrivial, primitive}\}.$$

Generalized Fermat Equations

$$S = \operatorname{Spec}(\mathbb{Z}[x,y,z]/(Ax^p + By^q - Cz^r)) \smallsetminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}}$$

The group $G = \mathbb{G}_m \cdot (\mu_p \times \mu_q \times \mu_r)$ acts on S with weights.

Generalized Fermat Equations

$$S = \operatorname{Spec}(\mathbb{Z}[x,y,z]/(Ax^p + By^q - Cz^r)) \smallsetminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}}$$

The group $G = \mathbb{G}_m \cdot (\mu_p \times \mu_q \times \mu_r)$ acts on S with weights.

We can form the curve $X = S/G$ whose points are exactly the equivalence classes of solutions:

$$X(\mathbb{Z}) = \{x, y, z \in R \mid Ax^p + By^q = Cz^r, \text{ nontriv., prim.}\}/\sim$$
$$\text{where } g \cdot (x,y,z) \sim (x,y,z).$$

Upside: these are easier to count than $S(\mathbb{Z})$.
Downside: the geometry of X is bad!

Generalized Fermat Equations

$$S = \mathrm{Spec}(\mathbb{Z}[x,y,z]/(Ax^p + By^q - Cz^r)) \smallsetminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}}$$

The group $G = \mathbb{G}_m \cdot (\mu_p \times \mu_q \times \mu_r)$ acts on S with weights.

We can form the **stacky** curve $\mathcal{X} = [S/G]$ whose points **remember the symmetries of each solution**:

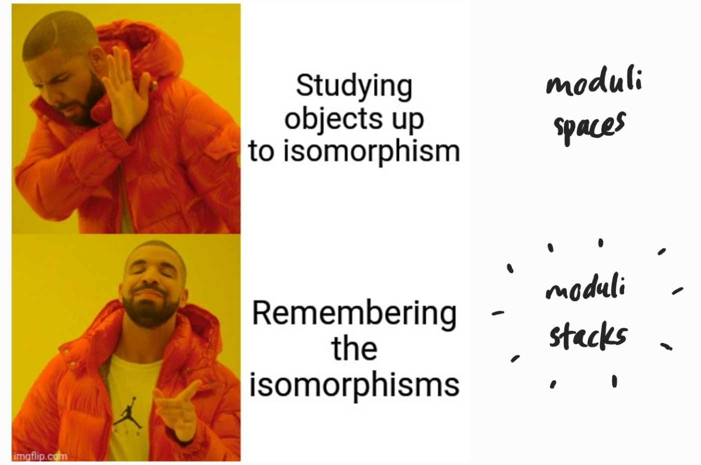
$\mathcal{X}(\mathbb{Z})$: objects: nontriv., prim. solutions to $Ax^p + By^q = Cz^r$

morphisms: $(x,y,z) \xrightarrow{g} g \cdot (x,y,z)$.

Upside: these are easier to count than $S(\mathbb{Z})$.
Downside: **none - stacks are awesome!**

Stacks

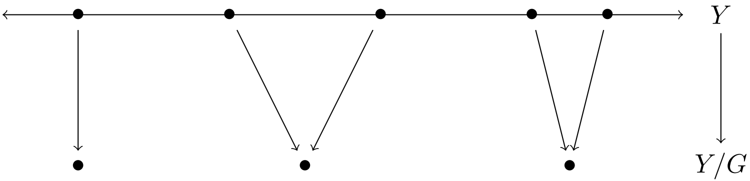
Rather than give a technical definition of a stack, here's a meme:



Stacks

Example

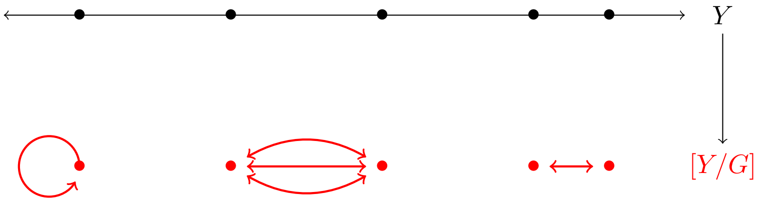
For a group G acting on a space Y, we can form the quotient space Y/G whose points are the equivalence classes of points under G:



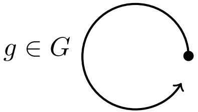
Stacks

Example

For a group G acting on a space Y, we can form the **quotient stack** $[Y/G]$ whose points are the **groupoid of G-orbits**:



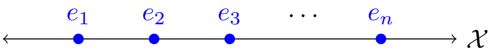
Special case: the classifying stack $[*/G] = BG$:



Stacky Curves

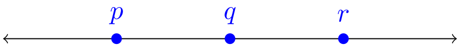
Here's an informal definition of a stacky curve:

A stacky curve $\mathcal{X}$ consists of an ordinary curve X, together with a finite number of marked points $P_1, \ldots, P_n$, each of which is decorated with a number $e_i =$ order of the group of symmetries of P_i.



Stacky Curves

Here's a cartoon of our stacky curve $[S/G]$, where $S =$ primitive integer solutions to $Ax^p + By^q = Cz^r$:



Generalized Fermat Equations, Revisited

To find solutions to $Ax^p + By^q = Cz^r$, we can exploit the geometry of $\mathcal{X} = [S/G]$:

Generalized Fermat Equations, Revisited

To find solutions to $Ax^p + By^q = Cz^r$, we can exploit the geometry of $\mathcal{X} = [S/G]$:

$$\begin{tikzcd} C_0 \arrow[d] \\ \mathcal{X} \end{tikzcd}$$

(1) Find a nice map $C_0 \to \mathcal{X}$ from a curve C_0 whose points are easy to find (e.g. a conic).

Generalized Fermat Equations, Revisited

To find solutions to $Ax^p + By^q = Cz^r$, we can exploit the geometry of $\mathcal{X} = [S/G]$:

$$\begin{tikzcd} C_1 \arrow[dr] & C_0 \arrow[l, dashed, "\text{twist}"'] \arrow[r, dashed, "\text{twist}"] \arrow[d] & C_2 \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

(1) Find a nice map $C_0 \to \mathcal{X}$ from a curve C_0 whose points are easy to find (e.g. a conic).

(2) Compute all twists of C_0 and their points.

Generalized Fermat Equations, Revisited

To find solutions to $Ax^p + By^q = Cz^r$, we can exploit the geometry of $\mathcal{X} = [S/G]$:

$$\begin{tikzcd}
C_1 \arrow[drr] & & C_0 \arrow[ll, dashed, "\text{twist}"'] \arrow[rr, dashed, "\text{twist}"] \arrow[d] & & C_2 \arrow[dll] \\
& & \mathcal{X} & &
\end{tikzcd}$$

(1) Find a nice map $C_0 \to \mathcal{X}$ from a curve C_0 whose points are easy to find (e.g. a conic).

(2) Compute all twists of C_0 and their points.

(3) Use descent to identify points on $\mathcal{X}$.

Generalized Fermat Equations, Revisited

Example

For $\mathcal{X} : x^2 + y^2 = z^2$, there is an étale map

$$\begin{array}{c} \mathbb{P}^1 \\ \big\downarrow \\ \mathcal{X} \end{array}$$

and $\mathbb{P}^1$ has infinitely many points which descend, so there are infinitely many primitive Pythagorean triples.

Generalized Fermat Equations, Revisited

Example (Poonen–Schaeffer–Stoll)

For $\mathcal{X} : x^2 + y^3 = z^7$, there is an étale map

$$\begin{tikzcd}
C_1 \arrow[drr] & & C \arrow[ll, dashed, "\text{twist}"'] \arrow[rr, dashed, "\text{twist}"] \arrow[d] & & C_2 \arrow[dll] & \cdots & C_{10} \arrow[dllll] \\
& & \mathcal{X} & & & &
\end{tikzcd}$$

where C is the Klein quartic, defined by $x^3y + y^3 + x = 0$. Descending points from C and its 10 twists gives 16 primitive solutions:

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad (0, \pm 1, \pm 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), \quad (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17).$$

Local-Global Principle for Algebraic Curves

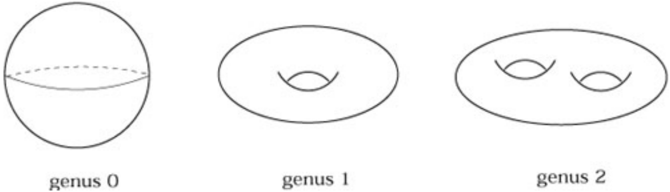
The classic local-global principle for an algebraic curve X asks if $X(\mathbb{Q}) \neq \varnothing$ is equivalent to $X(\mathbb{Q}_p) \neq \varnothing$ for all completions $\mathbb{Q}_p$, $p \leq \infty$.

Local-Global Principle for Algebraic Curves

The classic local-global principle for an algebraic curve X asks if $X(\mathbb{Q}) \neq \varnothing$ is equivalent to $X(\mathbb{Q}_p) \neq \varnothing$ for all completions $\mathbb{Q}_p$, $p \leq \infty$.

Let $g = g(X)$ be the genus of X. It is known that:

- (Hasse–Minkowski) If $g = 0$, the LGP holds for X.
- There are counterexamples to the LGP for all $g > 0$. For example, $X : 2y^2 = 1 - 17x^4$.



Local-Global Principle for Stacky Curves

For a stacky curve $\mathcal{X}$, we pose the *local-global principle for integral points*:

is $\mathcal{X}(\mathbb{Z}) \neq \varnothing$ equivalent to $\mathcal{X}(\mathbb{Z}_p) \neq \varnothing$ for all completions $\mathbb{Z}_p$?

Local-Global Principle for Stacky Curves

For a stacky curve $\mathcal{X}$, we pose the *local-global principle for integral points*:

is $\mathcal{X}(\mathbb{Z}) \neq \varnothing$ equivalent to $\mathcal{X}(\mathbb{Z}_p) \neq \varnothing$ for all completions $\mathbb{Z}_p$?

This time, the genus $g = g(\mathcal{X})$ can be *rational*:

$$g(\mathcal{X}) = g(X) + \frac{1}{2}\sum_{i=1}^{n} \frac{e_i - 1}{e_i}$$

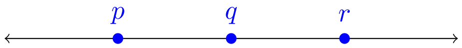
where X is the coarse space and $e_1, \ldots, e_n$ are the orders of the automorphisms groups at the finite number of stacky points.

When $\mathcal{X}$ is a *wild* stacky curve, I proved a more general formula for $g(\mathcal{X})$.

Local-Global Principle for Stacky Curves

Example

Our stacky curve $[S/G]$, where $S =$ primitive integer solutions to $Ax^p + By^q = Cz^r$, has genus $g = \frac{1}{2}\left(3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right)$.



For example, the $(2, 3, 7)$ curve has genus $g = \frac{85}{84}$.

This is the smallest possible genus for a 3-point stacky curve with $g > 1$.

Local-Global Principle for Stacky Curves

For $\mathcal{X} = [S/G]$ where $S : Ax^p + By^q = Cz^r$, $g = \frac{1}{2}\left(3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right)$.

Theorem (Bhargava–Poonen)

1. *If $g < \frac{1}{2}$, the LGP holds.*
2. *There are counterexamples to the LGP when $g = \frac{1}{2}$.*

Theorem (Darmon–Granville)

In the $(2, 2, n)$ case, with $g = \frac{n-1}{n}$, there are counterexamples to the LGP.

Joint work with Duque-Rosero, Keyes, Roy, Sankar, Wang (in progress): a complete solution in the $(2, 2, n)$ case.

Local-Global Principle for Stacky Curves

Almost Theorem (Duque-Rosero–Keyes–K.–Roy–Sankar–Wang)

Let $\mathcal{Y}$ be the stacky curve associated to the generalized Fermat equation $x^2 + By^2 = Cz^n$, with n odd, and set $K = \mathbb{Q}\left(\sqrt{-B}\right)$ and $R = \mathbb{Z}\left[\frac{1}{2nC}\right]$. Assuming $\mathrm{Cl}(K) = \{1\}$,

1. There is a finite list of conics $\{\mathcal{C}'_d\}$ of the form $\mathcal{C}'_d = \{uv = Cdw\} \subseteq \mathbb{P}(1,1,2)$ indexed by $d \in H^1(R, \mu_n)$ admitting étale covers $\pi_d : \mathcal{C}'_d \to \mathcal{Y}_K$ such that
$$\mathcal{Y}_K(R_K) = \coprod_{d \in H^1(R,\mu_n)} \pi_d(\mathcal{C}'_d(R_K)).$$
2. The local-global principle for R-integral points holds for $\mathcal{Y}$ if and only if there exists $d \in H^1(R, \mu_n)$ and a point $[u : v : w] \in \mathcal{C}'_d(R_K)$ such that $\frac{u^n}{dC^{(n-1)/2}}$ and $\frac{v^n}{d^{n-1}C^{(n-1)/2}}$ are Galois conjugates (under the action of $\mathrm{Gal}(K/\mathbb{Q})$).

Local-Global Principle for Stacky Curves

Almost Theorem (Duque-Rosero–Keyes–K.–Roy–Sankar–Wang)

Let $\mathcal{Y}$ be the stacky curve associated to the generalized Fermat equation $x^2 + By^2 = Cz^n$. Assuming $\mathrm{Cl}(K) = \{1\}$,

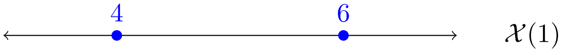
1. There is a finite list of conics $\{\mathbb{C}'_d\}$ capturing the R_K-points of $\mathcal{Y}_K$.
2. The local-global principle for R-integral points on $\mathcal{Y}$ is encoded in the R_K-points of $\{C'_d\}$.

Other goals:

- Remove the $\mathrm{Cl}(K) = \{1\}$ condition. (Done)
- Replace $x^2 + By^2$ with an arbitrary integral quadratic form $q(x, y)$.
- Adapt this strategy to spherical generalized Fermat equations with coefficients, e.g. $Ax^2 + By^3 = Cz^5$.

Another Example of a Stacky Curve

Here's another important stacky curve:



Fact: $\mathcal{X}(1) \cong \overline{\mathcal{M}}_{1,1}$, the compactified moduli stack of elliptic curves.

Fact 2: Modular curves give rise to *modular forms*.

Modular Forms

Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$.

A **modular form** of weight $2k$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

1. For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $f(z) = (cz + d)^{-2k} f(gz)$.
2. f is holomorphic at ∞.

Modular Forms

Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$.

A **modular form** of weight $2k$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

1. For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $f(z) = (cz+d)^{-2k} f(gz)$.
2. f is holomorphic at ∞.

Informal version: modular forms are highly symmetric holomorphic functions on the upper half-plane in $\mathbb{C}$.

Modular Forms

Given a modular form $f : \mathfrak{h} \to \mathbb{C}$, we can define a differential form $\omega = f(z)\,dz^k$.

By the symmetry of f, ω is not just defined on the upper half-plane, but on the quotient $\mathfrak{h}/SL_2(\mathbb{Z})$.

Compactifying by adding a point at ∞, this quotient $\overline{\mathfrak{h}/SL_2(\mathbb{Z})}$ becomes isomorphic to $\mathcal{X}(1)$, the moduli stack of elliptic curves.

Upshot: modular forms act like “functions” on the moduli stack $\mathcal{X}(1)$.

This allows one to define modular forms over any field K, as differential forms on the moduli stack $\mathcal{X}(1)$ of elliptic curves over K.

Modular Forms Mod p

Joint work with D. Zureick-Brown (in progress): describe the space of mod p modular forms using the stacky structure of $\mathcal{X}(1)$ and other modular curves over $\mathbb{F}_p$.

Modular Forms Mod p

Joint work with D. Zureick-Brown (in progress): describe the space of mod p modular forms using the stacky structure of $\mathcal{X}(1)$ and other modular curves over $\mathbb{F}_p$.

For $p > 3$, the story for $\mathcal{X}(1)$ is the same as over $\mathbb{C}$:

and $\bigoplus \mathcal{M}_k \cong \mathbb{F}_p[x_4, x_6]$ (originally due to Deligne, Edixhoven).

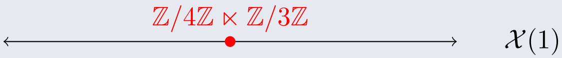
However, over $\mathbb{F}_2$ and $\mathbb{F}_3$, the stacky structure of $\mathcal{X}(1)$ looks different:

Modular Forms Mod 3

Joint work with D. Zureick-Brown (in progress): describe the space of mod p modular forms using the stacky structure of $\mathcal{X}(1)$ and other modular curves over $\mathbb{F}_p$.

Theorem (K.–Zureick-Brown 2024+ϵ)

For the **wild** *stacky curve* $\mathcal{X}(1)$ *over* $\mathbb{F}_3$,



the ring of modular forms is $\bigoplus \mathcal{M}_k \cong \mathbb{F}_3[x_2, x_{12}]$.

Modular Forms Mod 2

Joint work with D. Zureick-Brown (in progress): describe the space of mod p modular forms using the stacky structure of $\mathcal{X}(1)$ and other modular curves over $\mathbb{F}_p$.

Theorem (K.–Zureick-Brown 2024+ϵ)

For the **wild** *stacky curve* $\mathcal{X}(1)$ *over* $\mathbb{F}_2$,

$\mathbb{Z}/3\mathbb{Z} \ltimes Q_8$

$\mathcal{X}(1)$

the ring of modular forms is $\bigoplus \mathcal{M}_k \cong \mathbb{F}_2[x_1, x_{12}]$.

Thank you!

Questions?